GEOMETRIC TOMOGRAPHY WITH TOPOLOGICAL GUARANTEES

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Abstract. — We consider the problem of reconstructing a compact 3-manifold (with boundary) embedded in $\mathbb{R}^3$ from its cross-sections $\mathcal{S}$ with a given set of cutting planes $\mathcal{P}$ having arbitrary orientations. Using the obvious fact that a point $x \in \mathcal{P}$ belongs to the original object if and only if it belongs to $\mathcal{S}$, we follow a very natural reconstruction strategy: we say that a point $x \in \mathbb{R}^3$ belongs to the reconstructed object if (at least one of) its nearest point(s) in $\mathcal{P}$ belongs to $\mathcal{S}$. This coincides with the algorithm presented by Liu et al. in [**LBD+08**]. In the present paper, we prove that under appropriate sampling conditions, the output of this algorithm preserves the homotopy type of the original object. Using the homotopy equivalence, we also show that the reconstructed object is homeomorphic (and isotopic) to the original object. This is the first time that 3-dimensional shape reconstruction from cross-sections comes with theoretical guarantees.

1. Introduction

Overview. — This paper deals with the reconstruction of 3-dimensional geometric shapes from unorganized planar cross-sections. The need for such reconstructions is a result of the advances in medical imaging technology, especially in ultrasound tomography. In this context, the purpose is to construct a 3D model of an organ from a collection of ultrasonic images. When the images are provided by free-hand ultrasound devices, the cross-sections of the organ belong to planes that are not necessarily parallel. However, it is only very recently that reconstruction from unorganized cross-sections has been considered: A very first work by Payne and Toga [**PT94**] was restricted to easy cases of reconstruction that do not require branching between sections. In [**BG93**], Boissonnat and Geiger proposed a Delaunay-based algorithm for the case of serial planes, that has been generalized to arbitrarily oriented planes in [**DP97**] and [**BM07**]. Some more recent work [**JWC+05, LBD+08**] can handle the case of multilabel sections (multiple materials). Barequet and Vaxman's work [**BV09**] extends the work of [**LBD+08**] and can handle the case where the sections are only partially known.

Most of previous work has been restricted to the case of parallel cross-sections and is mostly based on the simple idea of connecting two sections if their orthogonal projections overlap.

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This paper analyzes a natural generalization of this idea for the case of non parallel sections, that has been proposed by Liu et al. in [**LBD$^+$08**]. We prove that under appropriate sampling conditions, the connection between the sections provided by this generalization is coherent with the connectivity structure of the object and the proposed reconstructed object is homeomorphic to the object. To the best of our knowledge, this work is the first to provide such a topological study in shape modeling from planar cross-sectional data. The only existing results studying the topology of the reconstructed object are restricted to the 2D variant of the problem ([**SBG06**] and [**MB08**]). We study the 2D problem in Section 2.4 of this paper.

Reconstruction Problem. — Let $\mathcal{O} \subset \mathbb{R}^3$ be a compact 3-manifold with boundary (denoted by $\partial\mathcal{O}$) of class C^2 [(1)]. The manifold $\mathcal{O}$ is cut by a set $\mathcal{P}$ of so-called cutting planes that are supposed to be in general position in the sense that none of these cutting planes are tangent to $\partial\mathcal{O}$. For any cutting plane $P \in \mathcal{P}$, we are given the set of intersection $\mathcal{O} \cap P$. There is no assumption on the geometry or the topology of these intersections. The goal is to reconstruct $\mathcal{O}$ from the given set of intersections denoted by $\mathcal{S}$.

Methodology. — We know that a point $x \in \mathcal{P}$ belongs to the original object if and only if it belongs to $\mathcal{S}$. The goal is now to determine whether a point $x \in \mathbb{R}^3$ belongs to $\mathcal{O}$ or not. We follow a very natural reconstruction strategy:

> We say that a point $x \in \mathbb{R}^3$ belongs to the reconstructed object if (at least one of) its **nearest point(s)** in $\mathcal{P}$ belongs to $\mathcal{S}$.

Different distance function (from $\mathcal{P}$) may be used in order to satisfy properties of interest for different applications (for example, to promote the connection between sections in the case of sparse data, or to impose a favorite direction to connect the sections, etc). A natural idea is to use the Euclidean distance. In this case, the reconstructed object coincides with the method introduced by Liu et al. in [**LBD$^+$08**]. We will analyze this method and present appropriate sampling conditions providing topological guarantees for the resulting reconstructed object.

Organization of the paper. — After this brief introduction, in Section 2 we provide a detailed description of the reconstructed object $\mathcal{R}$. The rest of the paper will be then devoted to prove that in the general case, under two appropriate sampling conditions, $\mathcal{R}$ and $\mathcal{O}$ are homotopy equivalent, and are more strongly homeomorphic. Indeed, the first sampling condition, called the *Separation Condition*, discussed in Section 2.1, ensures good connectivity between the sections, but does not necessarily imply the homotopy equivalence.

As we will see, in order to ensure the homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$, a second so-called *Intersection Condition* is required, c.f., Section 3.4. To make the connection between the upcoming sections more clear, in Section 3.1 we shortly outline the general strategy employed in proving the homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$. In Section 3.6, we provide a set of properties on the sampling of cutting planes to ensure the Separation and the Intersection Conditions. Finally, in Section 3.8 we show that the two shapes $\mathcal{O}$ and $\mathcal{R}$ are indeed homeomorphic (and even isotopic). Some preliminary notions of homotopy theory we use here are recalled in Appendix A.

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(1) This double-smoothness assumption is to simplify a technical part of the homotopy equivalence proof in this paper. Indeed, the same topological guarantees hold for a more general case where $\partial\mathcal{O}$ is of class $C^{1,1}$, i.e., when it is continuously differentiable and the normals satisfy a Lipschitz condition.

2. Characterization of the Reconstructed Object and First Condition

As said in the methodology section, we define the reconstructed object $\mathcal{R}$ as the set of points $x \in \mathbb{R}^3$ so that (at least one of) the **nearest point(s)** in $\mathcal{P}$ to x belongs to $\mathcal{S}$. This is based on the distance from the set of cutting planes $\mathcal{P}$. Since we consider the Euclidean distance, this involves the *arrangement of cutting planes*, i.e., the subdivision of $\mathbb{R}^3$ into convex polyhedral cells induced by the cutting planes. If a point $x \in \mathbb{R}^3$ belongs to a cell $\mathcal{C}$ of this arrangement, then its nearest point(s) in $\mathcal{P}$ belong(s) to the boundary of $\mathcal{C}$. Thus, we can decompose the problem into several subproblems as follows.

We can restrict our attention to a cell of the arrangement and reduce the reconstruction of $\mathcal{O}$ to the reconstruction of $\mathcal{O}_\mathcal{C} := \mathcal{O} \cap \mathcal{C}$ for all cells $\mathcal{C}$ of the arrangement.

Reconstructed Object in a Cell of the Arrangement. — We now focus on a cell $\mathcal{C}$ of the arrangement and describe the reconstructed object $\mathcal{R}_\mathcal{C}$ in $\mathcal{C}$. On each face f of $\mathcal{C}$, the intersection of the object $\mathcal{O}$ with f is given and consists of a set of connected regions called *sections*. By definition, the sections of a face of $\mathcal{C}$ are disjoint. However, two sections (on two neighbor faces of $\mathcal{C}$) may intersect along the intersection between their two corresponding faces. The boundary of a section A is denoted by ∂A and is a set of closed curves, called *section-contours*, that may be nested.

Let us write $\partial\mathcal{C}$ for the boundary of $\mathcal{C}$, and $\mathcal{F}_\mathcal{C}$ for the set of faces of $\mathcal{C}$. In the sequel, $\mathcal{S}_\mathcal{C}$ denotes the union of sections of all the faces of $\mathcal{C}$ and a point of $\mathcal{S}_\mathcal{C}$ is called a *section-point*. By the definition of $\mathcal{R}$, we have:

> A point $x \in \mathcal{C}$ is in the reconstructed object $\mathcal{R}_\mathcal{C}$ if one of its **nearest points** in $\partial\mathcal{C}$ is in $\mathcal{S}_\mathcal{C}$.

The definition of the reconstructed object in a cell $\mathcal{C}$ of the arrangement of the cutting planes naturally involves the Voronoi diagram of the faces of $\mathcal{C}$ defined as follows.

Voronoi Diagram of a Cell. — For a face f of $\mathcal{C}$, the Voronoi cell of f, denoted by $\mathrm{Vor}_\mathcal{C}(f)$, is defined as the set of all points in $\mathcal{C}$ that have f as the nearest face of $\mathcal{C}$, i.e.,

$$\mathrm{Vor}_\mathcal{C}(f) = \{\ x \in \mathcal{C} \mid d(x, f) \leq d(x, f'),\ \forall \text{ face } f' \text{ of } \mathcal{C}\ \}.$$

Here $d(.,.)$ stands for the Euclidean distance. The collection of all $\mathrm{Vor}_\mathcal{C}(f)$ when f runs over all the faces of $\mathcal{C}$ forms a tiling of $\mathcal{C}$, called the *Voronoi diagram of* $\mathcal{C}$.

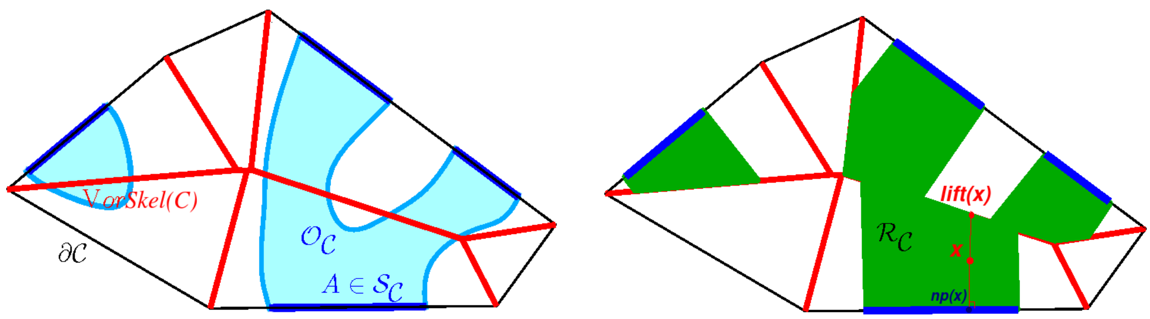

FIGURE 1. A 2D illustration of the partition of a cell $\mathcal{C}$ by the Voronoi Skeleton $\mathrm{VorSkel}(\mathcal{C})$. Left) The original shape $\mathcal{O}_\mathcal{C}$. Right) The reconstructed object $\mathcal{R}_\mathcal{C}$.

We write $\partial\mathrm{Vor}_{\mathcal{C}}(f)$ for the boundary of $\mathrm{Vor}_{\mathcal{C}}(f)$. The union of $\partial\mathrm{Vor}_{\mathcal{C}}(f)$ for all the faces f of $\mathcal{C}$ is called the *Voronoi Skeleton of* $\mathcal{C}$, and is denoted by $\mathrm{VorSkel}(\mathcal{C})$. $\mathrm{VorSkel}(\mathcal{C})$ is also called *the medial axis of the cell*, and is the locus of all the points in $\mathcal{C}$ that are at the same distance from at least two faces of $\mathcal{C}$. To simplify notation, when the cell $\mathcal{C}$ is understood from the context, we simply remove the index $\mathcal{C}$ and write $\mathrm{Vor}(f)$, $\partial\mathrm{Vor}(f)$, etc.

Definition 1 **(Nearest Point Characterization)**. — For any point x in $\mathcal{C}$, the *nearest point* in $\partial\mathcal{C}$ to x is the orthogonal projection of x onto the nearest face f of $\mathcal{C}$. This projection is denoted by $np_f(x)$. The set of all nearest points to x in $\partial\mathcal{C}$ is denoted by $Np_{\mathcal{C}}(x)$. Note that for any $x \notin \mathrm{VorSkel}(\mathcal{C})$, $Np_{\mathcal{C}}(x)$ is reduced to a single point. Based on this, and to simplify the presentation, sometimes we drop the index f, and by $np(x)$ we denote a point of $Np_{\mathcal{C}}(x)$.

Definition 2 **(Lift Function)**. — Let $x \in \mathcal{C}$ be a point in the Voronoi cell of a face f of $\mathcal{C}$. The *lift of x in $\mathcal{C}$*, denoted by $\mathrm{lift}_{\mathcal{C}}(x)$ (or simply $\mathrm{lift}(x)$ if $\mathcal{C}$ is trivially implied), is defined to be the unique point of $\partial\mathrm{Vor}_{\mathcal{C}}(f)$ such that the line defined by the segment $[x, \mathrm{lift}(x)]$ is orthogonal to f. In other words, $\mathrm{lift}(x)$ is the unique point in $\partial\mathrm{Vor}_{\mathcal{C}}(f)$ whose orthogonal projection onto f is $np(x)$.
— The *lift of a set of points* $X \subseteq \mathcal{C}$, denoted by $\mathrm{lift}(X)$, is the set of all the points $\mathrm{lift}(x)$ for $x \in X$, i.e., $\mathrm{lift}(X) := \{\, \mathrm{lift}(x) \,|\, x \in X \,\}$.
— The function $\mathcal{L} : \mathcal{C} \to \mathrm{VorSkel}(\mathcal{C})$ that maps each point $x \in \mathcal{C}$ to its lift in $\mathrm{VorSkel}(\mathcal{C})$ will be called the *lift function* in the sequel. For any $Y \subset \mathrm{VorSkel}(\mathcal{C})$, $\mathcal{L}^{-1}(Y)$ denotes the set of points $x \in \mathcal{C}$ such that $\mathrm{lift}(x) = y$ for some $y \in Y$.

We now present a geometric characterization of the reconstructed object using the described lifting procedure.

Characterization of the Reconstructed Object $\mathcal{R}_{\mathcal{C}}$. — If $\mathcal{S}_{\mathcal{C}} = \emptyset$, then, as we said before, for any point $x \in \mathcal{C}$, $np(x) \notin \mathcal{S}_{\mathcal{C}}$ and so $\mathcal{R}_{\mathcal{C}}$ is empty. Otherwise, let $A \in \mathcal{S}_{\mathcal{C}}$ be a section lying on a face of $\mathcal{C}$. For each point $a \in A$, the locus of all the points $x \in \mathcal{C}$ that have a as their nearest point in $\partial\mathcal{C}$ is the line segment $[a, \mathrm{lift}(a)]$ joining a to its lift. Therefore, the reconstructed object $\mathcal{R}_{\mathcal{C}}$ is the union of all the line-segments $[a, \mathrm{lift}(a)]$ for a point a in a section $A \in \mathcal{S}_{\mathcal{C}}$, i.e.,

$$\mathcal{R}_{\mathcal{C}} = \bigcup_{A \in \mathcal{S}_{\mathcal{C}}} \bigcup_{a \in A} [a, \mathrm{lift}(a)] = \mathcal{L}^{-1}(\mathrm{lift}(\mathcal{S}_{\mathcal{C}})).$$

Note that according to this characterization, if the lifts of two sections intersect in $\mathrm{VorSkel}(\mathcal{C})$, then these two sections are connected in $\mathcal{R}_{\mathcal{C}}$. This generalizes the classical overlapping criterion for the case of parallel cutting planes. The union of all the pieces $\mathcal{R}_{\mathcal{C}}$ over all cells $\mathcal{C}$ will be the overall reconstructed object $\mathcal{R}$. This definition is what has been proposed by Liu et al. in [**LBD+08**].

The rest of the paper is devoted to prove that under two appropriate sampling conditions, $\mathcal{R}$ and $\mathcal{O}$ are homotopy equivalent, and are indeed homeomorphic (and isotopic).

Let us first infer the following simple observation from the described characterization of $\mathcal{R}_{\mathcal{C}}$.

Proposition 1. — *The lift function $\mathcal{L} : \mathcal{R}_{\mathcal{C}} \to \mathrm{lift}(\mathcal{S}_{\mathcal{C}})$ is a homotopy equivalence.*

This is inferred trivially from the fact that the lift function retracts each segment $[a, \mathrm{lift}(a)]$ onto $\mathrm{lift}(a)$ continuously. See Figure 2 for an example.

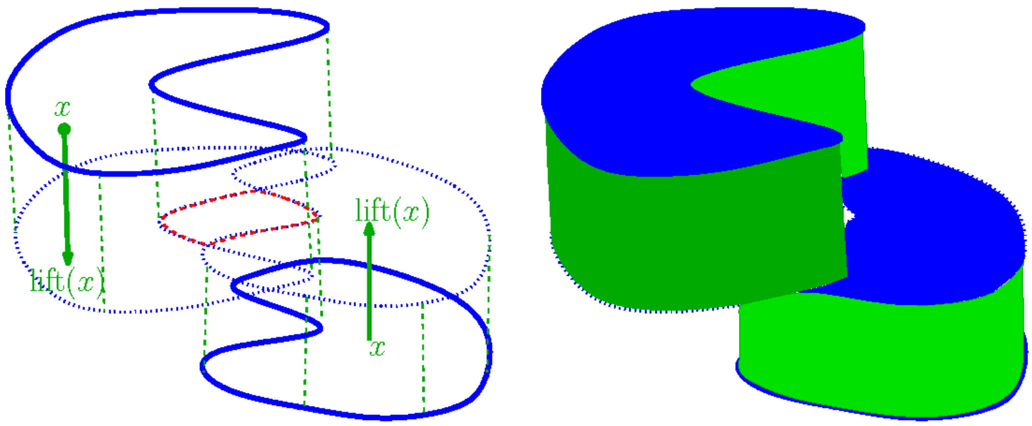

FIGURE 2. A 3D reconstruction example from a pair of parallel sections (in blue). The lift function retracts $\mathcal{R}_{\mathcal{C}}$ (in green) onto the lift of the sections.

2.1. First Sampling Condition: Separation Condition. — In this section, we provide the first sampling condition, under which the connection between the sections in the reconstructed object $\mathcal{R}$ will be the same as in the original object $\mathcal{O}$. Our discussion will be essentially based on the study of the *medial axis*, that we define now.

***Definition 3* (Medial Axis of $\partial\mathcal{O}$).** — Consider $\partial\mathcal{O}$ as a 2-manifold without boundary embedded in $\mathbb{R}^3$. The medial axis of $\partial\mathcal{O}$ denoted by $\mathrm{MA}(\partial\mathcal{O})$ contains two different parts: the so-called *internal* part denoted by $\mathrm{MA}_i(\partial\mathcal{O})$, which lies in $\mathcal{O}$, and the so-called *external* part denoted by $\mathrm{MA}_e(\partial\mathcal{O})$, which lies in $\mathbb{R}^3 \setminus \mathcal{O}$.

The *internal retract* $m_i : \partial\mathcal{O} \to \mathrm{MA}_i(\partial\mathcal{O})$ is defined as follows: for a point $x \in \partial\mathcal{O}$, $m_i(x)$ is the center of the maximum ball entirely included in $\mathcal{O}$ which passes through x. For any $x \in \partial\mathcal{O}$, $m_i(x)$ is unique. Symmetrically, we define *the external retract* $m_e : \partial\mathcal{O} \to \mathrm{MA}_e(\partial\mathcal{O})$: for a point $x \in \partial\mathcal{O}$, $m_e(x)$ is the center of the maximum ball entirely included in $\mathbb{R}^3 \setminus \mathcal{O}$ which passes through x. For any $x \in \partial\mathcal{O}$, $m_e(x)$ is unique but may be at infinity. In the sequel, we may write $m(a)$ for a point in $\{m_i(a), m_e(a)\}$.

The interesting point is that as discussed below if the sample of cutting planes is sufficiently dense, then the internal part of $\mathrm{MA}(\partial\mathcal{O})$ lies inside the defined reconstructed object and the external part of this medial axis lies outside the reconstructed object.

***Definition 4* (Separation Condition).** — We say that the set of cutting planes verifies the Separation Condition if

$$\mathrm{MA}_i(\partial\mathcal{O}) \subset \mathcal{R} \ \text{ and } \ \mathrm{MA}_e(\partial\mathcal{O}) \subset \mathbb{R}^3 \setminus \mathcal{R}\,.$$

In other words, *$\partial\mathcal{R}$ separates the internal and the external parts of the medial axis of $\partial\mathcal{O}$.* (That is where the name comes from.)

We will show that in each cell $\mathcal{C}$, the Separation Condition implies that $\partial\mathcal{R}_{\mathcal{C}}$ separates the internal and the external parts of the medial axis of $\partial\mathcal{O}_{\mathcal{C}}$. In order to study the Separation Condition in a cell $\mathcal{C}$, we will need to consider the medial axis of $\mathcal{O}_{\mathcal{C}}$ denoted by $\mathrm{MA}_i(\partial\mathcal{O}_{\mathcal{C}})$, defined as follows.

***Definition 5* (Medial axes in a cell $\mathcal{C}$ of the arrangement)**
The medial axis of $\mathcal{O}_{\mathcal{C}}$, $\mathrm{MA}_i(\partial\mathcal{O}_{\mathcal{C}})$, is defined as the set of all points in $\mathcal{O}_{\mathcal{C}}$ with at least two closest points in $\partial\mathcal{O}_{\mathcal{C}}$, see Figure 3. Symmetrically, by $\mathrm{MA}_e(\partial\mathcal{O}_{\mathcal{C}})$ we denote the medial axis of the closure of $\mathcal{C} \setminus \mathcal{O}_{\mathcal{C}}$.

Note that the two sets $\mathrm{MA}_i(\partial\mathcal{O}_{\mathcal{C}})$ and $\mathrm{MA}_i(\partial\mathcal{O}) \cap \mathcal{C}$ may be different.

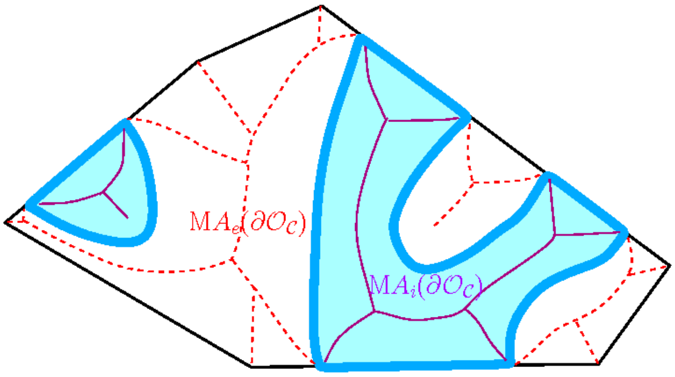

FIGURE 3. 2D example of medial axes in a cell $\mathcal{C}$ of the arrangement.

We also consider the *internal retract* $m_{i,\mathcal{C}} : \partial\mathcal{O}_{\mathcal{C}} \to \mathrm{MA}_i(\partial\mathcal{O}_{\mathcal{C}})$ defined as follows. For a point $x \in \partial\mathcal{O}_{\mathcal{C}}$, $m_{i,\mathcal{C}}(x)$ is the center of the maximum ball entirely included in $\mathcal{O}_{\mathcal{C}}$ which passes through x. Symmetrically, we can define *the external retract* $m_{e,\mathcal{C}} : \partial\mathcal{O}_{\mathcal{C}} \to \mathrm{MA}_e(\partial\mathcal{O}_{\mathcal{C}})$: for a point $x \in \partial\mathcal{O}_{\mathcal{C}}$, $m_{e,\mathcal{C}}(x)$ is the center of the maximum ball entirely included in $\mathcal{C} \setminus \mathcal{O}_{\mathcal{C}}$ which passes through x. It is easy to see that for any $x \in \partial\mathcal{O} \cap \mathcal{C}$, the segments $[x, m_{i,\mathcal{C}}(x)]$ and $[x, m_{e,\mathcal{C}}(x)]$ are subsegments of $[x, m_i(x)]$ and $[x, m_e(x)]$ respectively, and lie on the line defined by the normal to $\partial\mathcal{O}$ at x.

***Lemma 1* (Separation Condition Restricted to $\mathcal{C}$).** — *If the Separation Condition is verified, then $MA_i(\partial\mathcal{O}_{\mathcal{C}}) \subset \mathcal{R}_{\mathcal{C}}$ and $MA_e(\partial\mathcal{O}_{\mathcal{C}}) \subseteq \mathcal{C} \setminus \mathcal{R}_{\mathcal{C}}$.*

Proof. — We prove the first part, i.e., $\mathrm{MA}_i(\partial\mathcal{O}_{\mathcal{C}}) \subset \mathcal{R}_{\mathcal{C}}$. A similar proof gives the second part. Let m be a point in $\mathrm{MA}_i(\partial\mathcal{O}_{\mathcal{C}})$. Let $B(m)$ be the open ball centered at m which passes through the closest points to m in $\partial\mathcal{O}_{\mathcal{C}}$. Two cases can happen:

• Either, the closest points to m in $\partial\mathcal{O}_{\mathcal{C}}$ are in $\partial\mathcal{O}$, in which case m is a point in $\mathrm{MA}_i(\partial\mathcal{O})$. The Separation Condition states that $\mathrm{MA}_i(\partial\mathcal{O}) \subset \mathcal{R}$, and so $m \in \mathcal{R}_{\mathcal{C}} = \mathcal{R} \cap \mathcal{C}$.

• Otherwise, one of the closest points to m in $\partial\mathcal{O}_{\mathcal{C}}$ is a point a in some section $A \in \mathcal{S}_{\mathcal{C}}$. If a is on the boundary of A, then since along the section-contours $\partial\mathcal{O}_{\mathcal{C}}$ is non-smooth, a lies in $\mathrm{MA}_i(\partial\mathcal{O}_{\mathcal{C}})$ and coincides with m, and $m = a$ is trivially in $\mathcal{R}_{\mathcal{C}}$. Hence, we may assume that a lies in the interior of A. Therefore, the ball $B(m)$ is tangent to A at a, and the line segment $[a, m]$ is orthogonal to A. Since $B(m) \cap \partial\mathcal{C} = \emptyset$, m and a are in the same Voronoi cell of the Voronoi diagram of $\mathcal{C}$. Thus, $a \in \mathcal{S}_{\mathcal{C}}$ is the nearest point in $\partial\mathcal{C}$ to m. By the definition of $\mathcal{R}_{\mathcal{C}}$, we deduce that $m \in \mathcal{R}_{\mathcal{C}}$. □

Assume that the Separation Condition is verified. The first idea which comes to mind is to retract points of $\partial\mathcal{O}$ to $\partial\mathcal{R}$ by following the normal-directions. A point $x \in \partial\mathcal{O}$ which

lies outside $\mathcal{R}$ can move towards $m_i(x) \in \mathcal{R}$ and stop when $\partial\mathcal{R}$ is reached. A point $x \in \partial\mathcal{O}$ which lies inside $\mathcal{R}$, can move toward $m_e(x)$ and stop when $\partial\mathcal{R}$ is reached. According to a theorem by Wolter [**Wol92**], since $\partial\mathcal{O}$ is assumed to be of class C^2, this deformation will be a continuous retraction if each normal intersects $\partial\mathcal{R}$ at a single point. In such a case, $\partial\mathcal{O}$ can be deformed to $\partial\mathcal{R}$ homeomorphically. But a major problem is that $\mathcal{R}$ may have a complex shape (with cavities), so that a normal to $\partial\mathcal{O}$ intersects $\partial\mathcal{R}$ in several points. In such a case, such a retraction is not continuous and does not provide a deformation retract of $\mathcal{O}$ onto $\mathcal{R}$. However, we will be essentially following this intuitive idea by looking for a similar deformation retract of $\mathcal{O}$ onto a subshape of $\mathcal{R}$ (the so-called medial shape).

2.2. Guarantees on the Connections Between the Sections. — We now show that if the sample of cutting planes verifies the Separation Condition, then in each cell $\mathcal{C}$ of the arrangement, the connection between the sections is the same in $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$.

Theorem 1. — *If the sample of cutting planes verifies the Separation Condition, $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ induce the same connectivity components on the sections of $\mathcal{C}$.*

Proof. — The proof is given in two parts:

(I) **If two sections are connected in $\mathcal{O}_\mathcal{C}$, then they are connected in $\mathcal{R}_\mathcal{C}$.** Let A and A' be two sections in a same connected component K of $\mathcal{O}_\mathcal{C}$. Due to the non-smoothness of $\partial\mathcal{O}_\mathcal{C}$ at the boundary of the sections, ∂A and $\partial A'$ are contained in $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C})$. Thus, there is a path γ in $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C}) \cap K$ that connects a point $a \in \partial A$ to a point $a' \in \partial A'$. According to Lemma 1, $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C}) \subset \mathcal{R}_\mathcal{C}$. Thus, γ is a path in $\mathcal{R}_\mathcal{C}$ that connects A to A'.

(II) **If two sections are connected in $\mathcal{R}_\mathcal{C}$, then they are connected in $\mathcal{O}_\mathcal{C}$.** Let A and A' be two sections connected in $\mathcal{R}_\mathcal{C}$. Let γ be a path in $\mathcal{R}_\mathcal{C}$ that connects a point $a \in A$ to a point $a' \in A'$. For the sake of contradiction, suppose that a and a' are not in the same connected component of $\mathcal{O}_\mathcal{C}$. In this case, since γ joins two points in two different connected components of $\mathcal{O}_\mathcal{C}$, it must intersect $\mathrm{MA}_e(\partial\mathcal{O}_\mathcal{C})$. But this is a contradiction with the fact that $\gamma \subset \mathcal{R}_\mathcal{C}$, indeed, according to Lemma 1, we have $\mathrm{MA}_e(\partial\mathcal{O}_\mathcal{C}) \cap \mathcal{R}_\mathcal{C} = \emptyset$.

□

We state the following proposition that will be needed later in this section and in Section 3.

Proposition 2. — Under the Separation Condition, any connected component of $\partial\mathcal{O}$ is cut by at least one cutting plane.

Proof. — Suppose that K is a connected component of $\partial\mathcal{O}$ which is not cut by any cutting plane. There exists a cell $\mathcal{C}$ of the arrangement of hyperplanes such that one of the following two (symmetric) cases can happen: Either, there exists a connected component H of O which lies in the interior of $\mathcal{C}$ such that $K \subset \partial H$, Or, there exists a connected component H of the closure of $\mathbb{R}^3 \setminus \mathcal{O}$ which lies in the interior of $\mathcal{C}$ such that $K \subset \partial H$. Without loss of generality, suppose that ∂H bounds a connected component H of $\mathcal{O}$ in $\mathcal{C}$, H is entirely contained in the interior of $\mathcal{C}$, and $K \subset \partial H$, see Figure 4 (the other case follows similarly). Take a point m in the medial axis of H, i.e., $m \in H \cap \mathrm{MA}_i(\partial\mathcal{O})$. According to the Separation Condition, m belongs to $\mathcal{R}$. Thus, by the definition of $\mathcal{R}$, one of the nearest points of m in $\partial\mathcal{C}$, say $np(m)$, belongs to $\mathcal{S}$. Since H is not cut by any cutting plane, $H \cap \partial\mathcal{C}$ is empty and $np(m) \notin H$. Therefore, m and $np(m)$ are in two different connected components of $\mathcal{O}$, and the segment $[m, np(m)]$ should intersect $\mathrm{MA}_e(\partial\mathcal{O})$ at a point x. On the other hand, by the definition of

$\mathcal{R}$, the segment $[m, np(m)] \subset \mathcal{R}$. This contradicts the assumption of Separation Condition that $\mathcal{R} \cap \mathrm{MA}_e(\partial\mathcal{O}) = \emptyset$. □

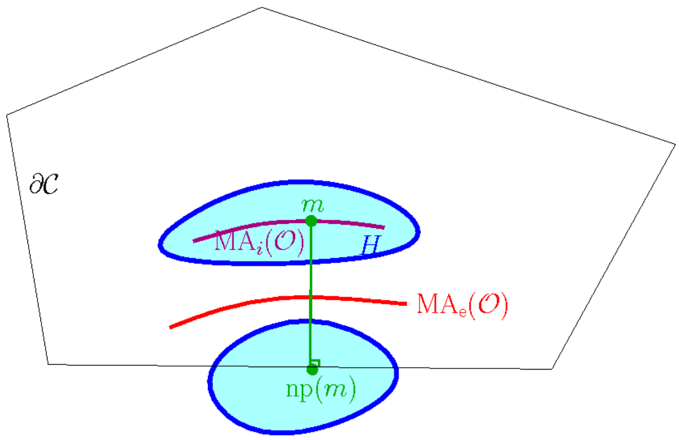

FIGURE 4. For the proof of Proposition 2.

2.3. How to Ensure the Separation Condition?— In this section we provide a sufficient condition for ensuring the Separation Condition. For this, we first need some definitions.

***Definition 6* (Reach).** — Let $\mathcal{O}$ be a connected compact 3-manifold with smooth boundary $\partial\mathcal{O}$ in $\mathbb{R}^3$. For $a \in \partial\mathcal{O}$, we define $\mathrm{reach}(a) = \min\big(d(a, m_i(a)), d(a, m_e(a))\big)$. The quantity $\mathrm{reach}(\mathcal{O})$ is defined as the minimum distance of $\partial\mathcal{O}$ from the medial axis of $\partial\mathcal{O}$:

$$\mathrm{reach}(\mathcal{O}) := \min_{m \in \mathrm{MA}(\partial\mathcal{O})} d(m, \partial\mathcal{O}) = \min_{a \in \partial\mathcal{O}} \mathrm{reach}(a).$$

Note that since $\mathcal{O}$ is compact and $\partial\mathcal{O}$ is of class C^2, $\mathrm{reach}(\mathcal{O})$ is strictly positive (see **[Fed59]** for a proof).

***Definition 7* (Reach restricted to a cell).** — Given a cell $\mathcal{C}$ of the arrangement, we define $\mathrm{reach}_{\mathcal{C}}(\mathcal{O}) = \min d(a, m(a))$, where either $a \in \partial\mathcal{O} \cap \mathcal{C}$ or $m(a) \in \mathrm{MA}(\partial\mathcal{O}) \cap \mathcal{C}$. By definition, we have $\mathrm{reach}(\mathcal{O}) = \min_{\mathcal{C}} (\mathrm{reach}_{\mathcal{C}}(\mathcal{O}))$.

***Definition 8* (Height of a Cell).** — Let $\mathcal{C}$ be a cell of the arrangement of the cutting planes. The *height* of $\mathcal{C}$, denoted by $h_{\mathcal{C}}$, is defined as the maximum distance of a point $x \in \mathcal{C}$ to its nearest point in the boundary of $\mathcal{C}$, see Figure 5. In other words, $h_{\mathcal{C}} := \max_{x \in \mathcal{C}} d(x, np(x))$.

We remark that the height of any cell $\mathcal{C}$ is at most half of the diagonal of $\mathcal{C}$. However, as the example of Figure 5 (right figure) shows, it may be much smaller than half of the diagonal. We now show that by bounding from above the height of the cells by a factor related to the reach of the object, we can ensure the Separation Condition.

***Lemma 2* (Sufficient Condition).** — *If the sample of cutting planes is sufficiently dense so that $h_{\mathcal{C}} < \mathrm{reach}_{\mathcal{C}}(\mathcal{O})$ for any cell $\mathcal{C}$ of the arrangement, then the Separation Condition is verified.*

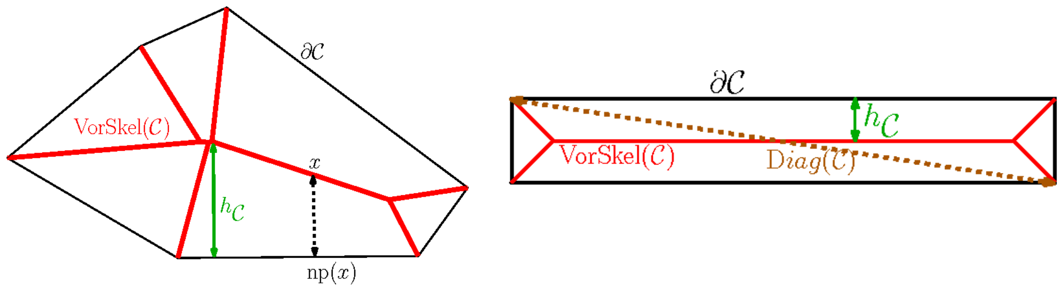

FIGURE 5. Left) Definition of the height of a cell $\mathcal{C}$. Right) Height of a cell $\mathcal{C}$ is bounded from above by half of the diagonal of $\mathcal{C}$, but may be much smaller for many configurations of cutting planes.

Proof. — The proof is straightforward. Let m be a point in $\mathrm{MA}(\partial\mathcal{O})$ in a cell $\mathcal{C}$ of the arrangement. We have

$$d(m, np(m)) \leq h_{\mathcal{C}} < \mathrm{reach}_{\mathcal{C}}(\mathcal{O}) \leq d(m, \partial\mathcal{O}).$$

Therefore, the Separation Condition is verified. □

2.4. Guarantees for 2D Shape Reconstruction from Line Cross-Sections. — We proved that under the Separation condition, the connectivity between the sections of $\mathcal{C}$ induced by the reconstructed object $\mathcal{R}_{\mathcal{C}}$ is coherent with the original shape $\mathcal{O}_{\mathcal{C}}$. We presented the proofs for the 3D case of the problem, but the proofs easily show that result is valid in any dimension. We show in this section that this is strong enough to imply the homotopy equivalence between $\mathcal{R}_{\mathcal{C}}$ and $\mathcal{O}_{\mathcal{C}}$ for the 2-dimensional variant of the reconstruction problem.

Consider the 2-dimensional variant of the reconstruction problem, that consists of reconstructing a 2D-shape from its intersections with arbitrarily oriented *cutting lines*. In this case the sections are line-segments.

We can focus on a cell $\mathcal{C}$ of the arrangement of the plane by the cutting lines. Similar definitions for the Voronoi diagram and the Voronoi skeleton of $\mathcal{C}$, the lift function and the reconstructed object $\mathcal{R}_{\mathcal{C}}$ can be considered. Under the Separation Condition, there is a bijection between the connected components of $\mathcal{R}_{\mathcal{C}}$ and $\mathcal{O}_{\mathcal{C}}$. By the definition of the reconstructed object, it is easy to see that any connected component of $\mathcal{R}_{\mathcal{C}}$ is a topological disk. On the other hand, according to Proposition 2, under the Separation Condition, any connected component of $\partial\mathcal{O}$ is cut by at least one cutting line. We easily deduce that any connected component of $\mathcal{O}_{\mathcal{C}}$ is a topological disk.

Using the sufficient condition presented in the last section, if for any cell $\mathcal{C}$ of the arrangement, $h_{\mathcal{C}} < \mathrm{reach}_{\mathcal{C}}(\mathcal{O})$, then the Separation Condition is ensured. We deduce the following theorem.

***Theorem 2* (Provably Good 2D Reconstruction).** — *If for any cell $\mathcal{C}$ of the arrangement of the cutting lines, $h_{\mathcal{C}} < reach_{\mathcal{C}}(\mathcal{O})$, then $\mathcal{R}$ is homeomorphic to $\mathcal{O}$.*

Proof. — According to Theorem 1, under the separation condition, there is a bijection between the connected components of $\mathcal{R}_{\mathcal{C}}$ and $\mathcal{O}_{\mathcal{C}}$. On the other hand, according to the discussion preceding the theorem, all the connected components of $\mathcal{O}_{\mathcal{C}}$ or $\mathcal{R}_{\mathcal{C}}$ are 2-dimensional disks. Therefore, there is a homotopy equivalence between each pair of corresponding connected

components of $\mathcal{O}_\mathcal{C}$ or $\mathcal{R}_\mathcal{C}$. This provides a homotopy equivalence between $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$. As we will explain in detail in Section 3.5, the homotopy equivalences in the different cells of the arrangement can be extended to a homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$. Finally, since $\mathcal{R}$ and $\mathcal{O}$ are two homotopy equivalent 2-dimensional submanifolds of $\mathbb{R}^2$, we deduce that there is a homeomorphism between $\mathcal{R}$ and $\mathcal{O}$. □

3. Topological Guarantees

To clarify the connection between the upcoming sections, let us shortly outline the general strategy employed in proving the homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$.

3.1. Proof Outline of the Homotopy Equivalence Between $\mathcal{R}$ and $\mathcal{O}$. — We will provide a homotopy equivalence between $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ in each cell of the arrangement. (And then glue these homotopy equivalences together to form a global homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$.) In Section 2.2 we showed that under the first sampling condition called Separation Condition the connection between the sections in the reconstructed object $\mathcal{R}_\mathcal{C}$ is the same as in $\mathcal{O}_\mathcal{C}$, in the sense that there is a bijection between the connected components of $\mathcal{R}_\mathcal{C}$ and the connected components of $\mathcal{O}_\mathcal{C}$. This implies that for proving the homotopy equivalence between $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$, it will be enough to show that the corresponding connected components have the same homotopy type. In order to extend these homotopy equivalences to a homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$, we will have to glue together the homotopy equivalences we obtain in the cells of the arrangement. This needs some care since the restriction to a section S of the two homotopy equivalences defined in the two adjacent cells of S may be different. To overcome this problem, we need to define an intermediate shape $\mathcal{M}_\mathcal{C}$ in each cell $\mathcal{C}$, called the *medial shape*. The medial shape has the following three properties:

- (i) The medial shape contains the sections of $\mathcal{C}$, i.e., $\mathcal{S}_\mathcal{C} \subseteq \mathcal{M}_\mathcal{C}$.
- (ii) There is a (strong) deformation retract r from $\mathcal{O}_\mathcal{C}$ to $\mathcal{M}_\mathcal{C}$. In particular, this map is a homotopy equivalence between $\mathcal{O}_\mathcal{C}$ and $\mathcal{M}_\mathcal{C}$. And its restriction to $\mathcal{S}_\mathcal{C}$ is the identity map.
- (iii) Under the first sampling condition (Separation Condition), $\mathcal{M}_\mathcal{C} \subseteq \mathcal{R}_\mathcal{C}$.

The first two properties will be crucial to guarantee that the homotopy equivalences conform on each section under the Separation Condition. Indeed, the map $\mathcal{O}_\mathcal{C} \to \mathcal{M}_\mathcal{C} \hookrightarrow \mathcal{R}_\mathcal{C}$, obtained by composing the deformation retract and the inclusion, restricts to the identity map on each section of $\mathcal{S}_\mathcal{C}$. Thus, we can glue all these maps to obtain a global map from $\mathcal{O}$ to $\mathcal{R}$.

Using a generalized version of the nerve theorem (see Section 3.5) and property (ii) above, we can then reduce the problem to proving that the inclusion $i : \mathcal{M}_\mathcal{C} \hookrightarrow \mathcal{R}_\mathcal{C}$ forms a homotopy equivalence in each cell. Using Whitehead's theorem, it will be enough to show that the inclusion i induces isomorphisms between the corresponding homotopy groups. Under the Separation Condition, we prove that i induces an injective map on the first homotopy groups, and that all higher dimensional homotopy groups of $\mathcal{M}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ are trivial. Unfortunately, the Separation Condition does not ensure in general the surjectivity of i on the first homotopy groups. To overcome this problem, we need to impose a second condition called *Intersection Condition.* Under the Intersection Condition, the map i will be surjective on the first homotopy groups, leading to a homotopy equivalence between $\mathcal{O}$ and $\mathcal{R}$.

According to the guarantees on the connectivity between the sections (Theorem 1), to prove the homotopy equivalence between $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ under the Separation Condition, we may restrict our attention to each of the corresponding connected components.

In the sequel, to simplify the notations and the presentation, we suppose that $\mathcal{O}_\mathcal{C}$ and thus $\mathcal{R}_\mathcal{C}$ are connected, and we show that $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ have the same homotopy type. It is clear that the same proofs can be applied to each corresponding connected components of $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ to imply the homotopy equivalence in the general case of multiple connected components.

3.2. Medial Shape. — We now define an intermediate shape in each cell $\mathcal{C}$ of the arrangement called the *medial shape*. The medial shape enjoys a certain number of important properties, discussed in this section, which makes it playing an important role in obtaining the homotopy equivalence between $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$.

Definition 9 **(Medial Shape $\mathcal{M}_\mathcal{C}$).** — Let x be a point in $\mathcal{S}_\mathcal{C} \subset \partial\mathcal{O}_\mathcal{C}$. Let $w(x) = [x, m_{i,\mathcal{C}}(x)]$ be the segment in the direction of the normal to $\partial\mathcal{O}_\mathcal{C}$ at x which connects x to the point $m_{i,\mathcal{C}}(x) \in \mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C})$. We add to $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C})$ all the segments $w(x)$ for all the points $x \in \mathcal{S}_\mathcal{C}$. We call the resulting shape $\mathcal{M}_\mathcal{C}$, see Figure 6 (Left) for an example. More precisely,

$$\mathcal{M}_\mathcal{C} := \mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C}) \cup \Big(\bigcup_{x \in \mathcal{S}_\mathcal{C}} w(x) \Big).$$

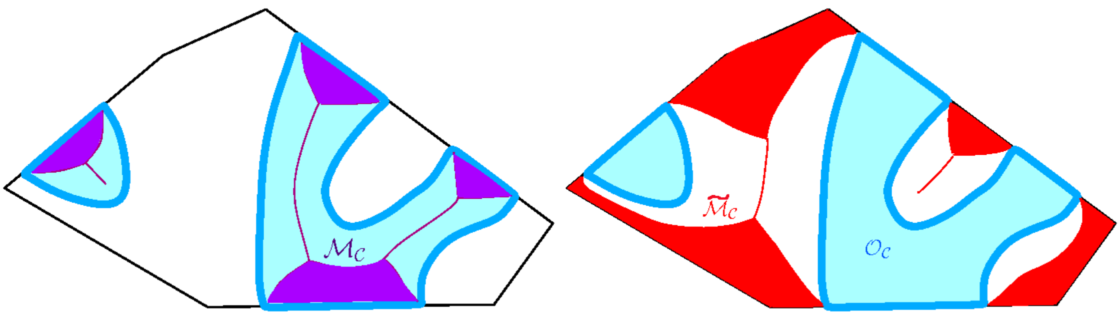

FIGURE 6. A 2D illustration of the medial shape $\mathcal{M}_\mathcal{C}$ in purple (left) and $\widetilde{\mathcal{M}}_\mathcal{C}$ in red (right).

Proposition 3. — *The medial shape verifies the following set of properties:*

(i) *The medial shape contains the sections of $\mathcal{C}$, i.e., $\mathcal{S}_\mathcal{C} \subseteq \mathcal{M}_\mathcal{C}$.*

(ii) *There is a (strong) deformation retract r from $\mathcal{O}_\mathcal{C}$ to $\mathcal{M}_\mathcal{C}$. In particular, this map is a homotopy equivalence between $\mathcal{O}_\mathcal{C}$ and $\mathcal{M}_\mathcal{C}$. And its restriction to $\mathcal{S}_\mathcal{C}$ is the identity map.*

(iii) *Under the Separation Condition, $\mathcal{M}_\mathcal{C} \subseteq \mathcal{R}_\mathcal{C}$.*

Proof. — (i) This property is true by the definition of the medial shape.

(ii) This is obtained by deforming $\mathcal{O}_\mathcal{C}$ to $\mathcal{M}_\mathcal{C}$ in the direction of the normals to the boundary $\partial\mathcal{O}_\mathcal{C}$. Note that the boundary $\partial\mathcal{O}_\mathcal{C}$ is smooth except on the boundaries of sections in $\mathcal{S}_\mathcal{C}$, and the boundaries of the sections in $\mathcal{S}_\mathcal{C}$ are already in $\mathcal{M}_\mathcal{C}$, thus, the deformation retract is well defined. Moreover, since $\partial\mathcal{O}$ (and so $\partial\mathcal{O}_\mathcal{C}$) is supposed to be of class C^2,

according to a theorem by Wolter [**Wol92**], this deformation is continuous and is hence a continuous deformation retract from $\mathcal{O}_\mathcal{C}$ to $\mathcal{M}_\mathcal{C}$ [(2)].

(*iii*) Since $\mathcal{M}_\mathcal{C} = \mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C}) \cup \left(\bigcup_{x\in\mathcal{S}_\mathcal{C}} w(x)\right)$ and in addition $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C}) \subset \mathcal{R}_\mathcal{C}$, it will be sufficient to show that for any x in a section $A \in \mathcal{S}_\mathcal{C}$, $w(x) \subset \mathcal{R}_\mathcal{C}$. (Recall that $w(x)$ is the orthogonal segment to $\partial\mathcal{O}_\mathcal{C}$ at x that joins x to the corresponding medial point $m_{i,\mathcal{C}}(x)$ in $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C})$.) We will show that $w(x)$ is contained in the segment $[x, \mathrm{lift}(x)]$. The point x is the closest point in $\partial\mathcal{O}_\mathcal{C}$ to $m_{i,\mathcal{C}}(x)$. Thus, the ball centered at $m_{i,\mathcal{C}}(a)$ and passing through x is entirely contained in $\mathcal{O}$ and its interior is empty of points of $\partial\mathcal{C}$. Thus, in the Voronoi diagram of $\mathcal{C}$, $m_{i,\mathcal{C}}(a)$ is in the same Voronoi cell as x. On the other hand, x is the closest point in $\mathcal{S}_\mathcal{C} \subset \partial\mathcal{O}_\mathcal{C}$ to $\mathrm{lift}(x)$. It easily follows that $d(x, \mathrm{lift}(x)) \geq d(x, m_{i,\mathcal{C}}(x))$. It follows that the segment $[x, m_{i,\mathcal{C}}(x)] = w(x)$ is a subsegment of $[x, \mathrm{lift}(x)]$. Therefore, by the definition of $\mathcal{R}_\mathcal{C}$, $w(x) \subset \mathcal{R}_\mathcal{C}$.

□

We end this section with the following important remark and proposition which will be used in the next section. By replacing the shape $\mathcal{O}_\mathcal{C}$ with its complementary set we may define an *exterior medial shape* $\widetilde{\mathcal{M}_\mathcal{C}}$. This is more precisely defined as follows. Let $\widetilde{\mathcal{O}}$ be the closure of the complementary set of $\mathcal{O}$ in $\mathbb{R}^3$. And let $\widetilde{\mathcal{O}_\mathcal{C}}$ be the intersection of $\widetilde{\mathcal{O}}$ with the cell $\mathcal{C}$. The medial shape of $\widetilde{\mathcal{O}_\mathcal{C}}$, denoted by $\widetilde{\mathcal{M}_\mathcal{C}}$, is the union of the medial shapes of the connected components of $\widetilde{\mathcal{O}_\mathcal{C}}$, see Figure 6 (Right). Similarly, under the Separation Condition, the following proposition holds.

Proposition 4. — *Let $\widetilde{\mathcal{O}_\mathcal{C}}$ be the closure of the complementary set of $\mathcal{O}_\mathcal{C}$ in $\mathcal{C}$ and $\widetilde{\mathcal{M}_\mathcal{C}}$ be the medial shape of $\widetilde{\mathcal{O}_\mathcal{C}}$. Under the Separation Condition:* (*i*) *There is a strong deformation retract from $\mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}}$ to $\mathcal{O}_\mathcal{C}$, and* (*ii*) *We have $\mathcal{R}_\mathcal{C} \subset \mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}}$.*

Proof. — The proof of Property (*i*) is similar to the proof of Proposition 3 by deforming along the normal vectors to the boundary of $\widetilde{\mathcal{O}_\mathcal{C}}$. The second property (*ii*) is equivalent to $\widetilde{\mathcal{M}_\mathcal{C}} \subset \mathcal{C} \setminus \mathcal{R}_\mathcal{C}$. □

3.3. Topological Guarantees Implied by the Separation Condition. — Throughout this section, we suppose that the Separation Condition holds. By the discussion at the end of Section 3.1, and without loss of generality, we may assume that $\mathcal{O}_\mathcal{C}$ and hence $\mathcal{R}_\mathcal{C}$ are connected. Thus, $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ are connected compact topological 3-manifolds embedded in $\mathbb{R}^3$.

In Section 3.2, we defined the medial shape $\mathcal{M}_\mathcal{C}$ and showed that $\mathcal{M}_\mathcal{C}$ is homotopy equivalent to $\mathcal{O}_\mathcal{C}$ by giving a (strong) deformation retract from $\mathcal{O}_\mathcal{C}$ to $\mathcal{M}_\mathcal{C}$. We have also shown that under the Separation Condition, $\mathcal{M}_\mathcal{C} \subset \mathcal{R}_\mathcal{C}$. Using these properties, the goal will be to prove that the inclusion $i : \mathcal{M}_\mathcal{C} \hookrightarrow \mathcal{R}_\mathcal{C}$ is a homotopy equivalence. As the objects we are manipulating are all CW-complexes, homotopy equivalence is equivalent to weak homotopy equivalence according to Whitehead's theorem.

Hence, it will be enough to show that $i : \mathcal{M}_\mathcal{C} \hookrightarrow \mathcal{R}_\mathcal{C}$ induces isomorphism between the corresponding homotopy groups.

[(2)] Indeed, it is possible to show that the same result holds for a more general case where $\partial\mathcal{O}$ is of class $C^{1,1}$ (i.e., it is continuously differentiable and its normal satisfies a Lipschitz condition). In that case, Wolter's theorem on the continuity is not valid anymore (i.e., there are counter examples). However, one can prove the existence of a continuous deformation in our spacial case by using Lieutier's deformation based on generalized distance function. We refer the interested reader to Lieutier's paper [**Lie04**] for more details.

Injectivity on the Level of Homotopy Groups. — We will make use of the lift function in $\mathcal{C}$, c.f. Section 2. We consider the restriction of the lift function to the reconstructed object $\mathcal{R}_\mathcal{C}$.

According to Proposition 1, the lift function $\mathcal{L} : \mathcal{R}_\mathcal{C} \to \text{lift}(\mathcal{S}_\mathcal{C})$ is a homotopy equivalence. We claim that it will be sufficient to show that the restriction of the lift function to $\mathcal{M}_\mathcal{C}$ is a weak homotopy equivalence. Because in this case, using the following diagram and Whitehead's theorem, we can infer that $i : \mathcal{M}_\mathcal{C} \hookrightarrow \mathcal{R}_\mathcal{C}$ is a homotopy equivalence.

$$\begin{tikzcd}
\mathcal{M}_\mathcal{C} \arrow[r, hook, "i"] \arrow[dr, "\mathcal{L}"'] & \mathcal{R}_\mathcal{C} \arrow[d, "\mathcal{L}"] \\
& \text{lift}(\mathcal{S}_\mathcal{C})
\end{tikzcd}$$

More precisely, if $\mathcal{L} : \mathcal{M}_\mathcal{C} \to \text{lift}(\mathcal{S}_\mathcal{C})$ is a weak homotopy equivalence (what we will prove below), since $\mathcal{L} : \mathcal{R}_\mathcal{C} \to \text{lift}(\mathcal{S}_\mathcal{C})$ is also a homotopy equivalence and because of the commutativity of the above diagram, the inclusion $i : \mathcal{M}_\mathcal{C} \hookrightarrow \mathcal{R}_\mathcal{C}$ induces isomorphisms between the homotopy groups of $\mathcal{M}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$. Thus, Whitehead's theorem implies that i is a homotopy equivalence. We first show that under the Separation Condition, the restricted lift function $\mathcal{L}_{|\mathcal{M}_\mathcal{C}}$ induces injections on the level of homotopy groups.

***Theorem 3* (Injectivity).** — *Under the Separation Condition, the homomorphisms between the homotopy groups of $\mathcal{M}_\mathcal{C}$ and* $\text{lift}(\mathcal{S}_\mathcal{C})$*, induced by the lift function $\mathcal{L}$, are injective.*

Proof. — Under the Separation Condition, we have $\mathcal{M}_\mathcal{C} \subset \mathcal{R}_\mathcal{C}$. Let $\widetilde{\mathcal{M}_\mathcal{C}}$ be the medial shape of the closure of the complementary set of $\mathcal{O}_\mathcal{C}$ in $\mathcal{C}$. We refer to the discussion at the end of the previous section for more details. Recall that by Proposition 4, we have $\mathcal{R}_\mathcal{C} \subset \mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}}$, and there exists a deformation retract from $\mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}}$ to $\mathcal{O}_\mathcal{C}$ (in particular $\mathcal{O}_\mathcal{C}$ and $\mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}}$ are homotopy equivalent). We have now the following commutative diagram in which every map (except the lift function $\mathcal{L}$) is an injection (or an isomorphism) on the level of homotopy groups.

$$\begin{tikzcd}
& \mathcal{O}_\mathcal{C} \arrow[dl, "\simeq"'] & \\
\mathcal{M}_\mathcal{C} \arrow[r, hook, "i"] \arrow[dr, "\mathcal{L}"'] & \mathcal{R}_\mathcal{C} \arrow[r, hook] \arrow[d, "\simeq"] & \mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}} \arrow[ul, "\simeq"'] \\
& \text{lift}(\mathcal{S}_\mathcal{C}) &
\end{tikzcd}$$

Using this diagram, the injectivity on the level of homotopy groups is clear: For any integer $j \geq 1$, consider the induced homomorphism $\mathcal{L}_* : \pi_j(\mathcal{M}_\mathcal{C}) \to \pi_j(\text{lift}(\mathcal{S}_\mathcal{C}))$. Let $x \in \pi_j(\mathcal{M}_\mathcal{C})$ be so that $\mathcal{L}_*(x)$ is the identity element of $\pi_j(\text{lift}(\mathcal{S}_\mathcal{C}))$. It is sufficient to show that x is the identity element of $\pi_j(\mathcal{M}_\mathcal{C})$. Following the maps of the diagram, and using the homotopy equivalence between $\text{lift}(\mathcal{S}_\mathcal{C})$ and $\mathcal{R}_\mathcal{C}$, we have that $i_*(x)$ is mapped to the identity element of $\pi_j(\mathcal{R}_\mathcal{C})$. Then, by the inclusion $\mathcal{R}_\mathcal{C} \hookrightarrow \mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}}$, it goes to the identity element of $\mathcal{C} \setminus \widetilde{\mathcal{M}_\mathcal{C}}$, and by the two retractions, it will be mapped to the identity element of $\pi_j(\mathcal{M}_\mathcal{C})$. As this diagram is commutative, we infer that x is the identity element of $\pi_j(\mathcal{M}_\mathcal{C})$. Thus, $\mathcal{L}_*$:

$\pi_j(\mathcal{M}_\mathcal{C}) \to \pi_j(\mathrm{lift}(\mathcal{S}_\mathcal{C}))$ is injective for all $j \geq 1$. The injectivity for $j = 0$ is already proved in Theorem 1. □

We have shown that under the Separation Condition, the lift function $\mathcal{L} : \mathcal{M}_\mathcal{C} \to \mathrm{lift}(\mathcal{S}_\mathcal{C})$ induces injective morphisms between the homotopy groups of $\mathcal{M}_\mathcal{C}$ and $\mathrm{lift}(\mathcal{S}_\mathcal{C})$. If these induced morphisms were surjective, then $\mathcal{L}$ would be a homotopy equivalence (by Whitehead's theorem). We will show below that the Separation Condition implies the surjectivity for all the homotopy groups except for dimension one (fundamental groups). Indeed, we will show that under the Separation Condition, all the i-dimensional homotopy groups of $\mathcal{M}_\mathcal{C}$ and $\mathrm{lift}(\mathcal{S}_\mathcal{C})$ for $i \geq 2$ are trivial. Once this is proved, it will be sufficient to study the surjectivity of $\mathcal{L}_* : \pi_1(\mathcal{M}_\mathcal{C}) \to \pi_1(\mathrm{lift}(\mathcal{S}_\mathcal{C}))$.

Remark 1. — Note that the injectivity in the general form above remains valid for the corresponding reconstruction problems in dimensions greater than three. However, the vanishing results on higher homotopy groups of $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ are only valid in dimensions two and three.

The topological structures of $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ are determined by their fundamental groups. — In this section, we show that if the Separation Condition is verified, then the topological structure of the portion of $\mathcal{O}$ in a cell $\mathcal{C}$ (i.e., $\mathcal{O}_\mathcal{C}$) is simple enough, in the sense that for all $i \geq 2$, the i-dimensional homotopy group of $\mathcal{O}_\mathcal{C}$ is trivial. We can easily show that $\mathcal{R}_\mathcal{C}$ has the same property [(3)]. As a consequence, the topological structures of $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ are determined by their fundamental group, $\pi_1(\mathcal{O}_\mathcal{C})$ and $\pi_1(\mathcal{R}_\mathcal{C})$.

We first state the following general theorem for an arbitrary embedded 3-manifold with connected boundary.

Theorem 4. — *Let K be a connected 3-manifold in $\mathbb{R}^3$ with a (non-empty) connected boundary. Then for all $i \geq 2$, $\pi_i(K) = \{0\}$.*

This theorem can be easily obtained from Sphere Theorem, see e.g., Corollary 3.9 of [**Hat02**] or for more details [**Mem10**]. From this theorem, we infer the two following theorems.

Theorem 5. — *Under the Separation Condition, $\pi_i(\mathcal{O}_\mathcal{C}) = \{0\}$, for all $i \geq 2$.*

Proof. — We only make use of the fact that under the Separation Condition, any connected component of $\partial\mathcal{O}$ is cut by at least one cutting plane (Proposition 2). In this case, every connected component of $\mathcal{O}_\mathcal{C}$ is a 3-manifold with connected boundary. The theorem follows as a corollary of Theorem 4. □

Theorem 6. — *$\pi_i(\mathcal{R}_\mathcal{C}) = \{0\}$, for all $i \geq 2$.*

Proof. — Using Theorem 4, it will be sufficient to show that the boundary of any connected component K of $\mathcal{R}_\mathcal{C}$ is connected. Let x and y be two points on the boundary of K, and let S and S' be two sections so that $x \in [a, \mathrm{lift}(a)]$ for some $a \in S$ and $y \in [b, \mathrm{lift}(b)]$ for some $b \in S'$. By the definition of $\mathcal{R}_\mathcal{C}$, x is connected to S in $\partial\mathcal{R}_\mathcal{C}$, and y is connected to S' in $\partial\mathcal{R}_\mathcal{C}$. On the other hand, since S and S' are two sections in the connected component K of $\mathcal{R}_\mathcal{C}$, they lie on ∂K and are connected to each other in ∂K (and so in $\partial\mathcal{R}_\mathcal{C}$). Thus, x is connected to y in $\partial\mathcal{R}_\mathcal{C}$. □

(3) Recall that for simplifying the presentation, we assume that $\mathcal{O}_\mathcal{C}$ and so $\mathcal{R}_\mathcal{C}$ are connected. The same proof shows that in the general case, the same property holds for each connected component of $\mathcal{O}_\mathcal{C}$ or $\mathcal{R}_\mathcal{C}$.

3.4. Second Condition: Intersection Condition. — In the previous section, we saw that under the Separation Condition, the topological structures of $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ are determined by their fundamental group $\pi_1(\mathcal{O}_\mathcal{C})$ and $\pi_1(\mathcal{R}_\mathcal{C})$, respectively. The goal of this section is to find a way to ensure an isomorphism between the fundamental groups of $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$. We recall that as $\mathcal{O}_\mathcal{C}$ and $\mathcal{M}_\mathcal{C}$ are homotopy equivalent, $\pi_1(\mathcal{O}_\mathcal{C})$ is isomorphic to $\pi_1(\mathcal{M}_\mathcal{C})$. On the other hand, $\mathcal{R}_\mathcal{C}$ and $\text{lift}(\mathcal{S}_\mathcal{C})$ are homotopy equivalent, and $\pi_1(\mathcal{R}_\mathcal{C})$ is isomorphic to $\pi_1(\text{lift}(\mathcal{S}_\mathcal{C}))$ (c.f. last diagram). Thus, it will be sufficient to compare $\pi_1(\mathcal{M}_\mathcal{C})$ and $\pi_1(\text{lift}(\mathcal{S}_\mathcal{C}))$.

We consider $\mathcal{L}_* : \pi_1(\mathcal{M}_\mathcal{C}) \to \pi_1(\text{lift}(\mathcal{S}_\mathcal{C}))$, the map induced by the lift function from $\mathcal{M}_\mathcal{C}$ to $\text{lift}(\mathcal{S}_\mathcal{C})$ on fundamental groups . We showed that $\mathcal{L}_*$ is injective. A natural condition to ensure that $\mathcal{L}_*$ is an isomorphism is to impose that $\text{lift}(\mathcal{S}_\mathcal{C})$ *is contractible* (or more generally, each connected component of $\text{lift}(\mathcal{S}_\mathcal{C})$ is contractible). This is very common in practice, where the sections are contractible and sufficiently close to each other. In this case, all the homotopy groups $\pi_j(\text{lift}(\mathcal{S}_\mathcal{C}))$ are trivial and by injectivity of $\mathcal{L}_*$ proved in the previous section, $\mathcal{L}_*$ becomes an isomorphism in each dimension. Hence, the homotopy equivalence between $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ can be deduced.

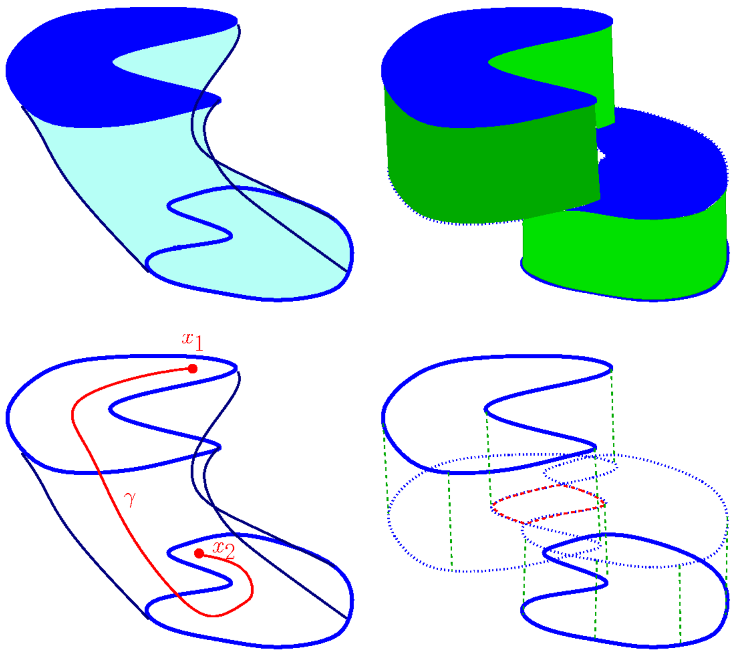

FIGURE 7. Left) Original shape. Right) Reconstructed shape. 3D example of the case where the lift function from $\mathcal{M}_\mathcal{C}$ to $\text{lift}(\mathcal{S}_\mathcal{C})$ fails to be surjective: x_1 and x_2 are two points with the same lift in $\text{lift}(\mathcal{S}_\mathcal{C})$. The lift of any curve γ connecting x_1 and x_2 in $\mathcal{M}_\mathcal{C}$ provides a non-zero element of $\pi_1(\text{lift}(\mathcal{S}_\mathcal{C}), x)$. The reconstructed shape is a torus and is not homotopy equivalent to the original shape which is a twisted cylinder.

However, the map $\mathcal{L}_* : \pi_1(\mathcal{M}_\mathcal{C}) \to \pi_1(\text{lift}(\mathcal{S}_\mathcal{C}))$ fails to be surjective in general (where the connected components of $\text{lift}(\mathcal{S}_\mathcal{C})$ are not necessarily contractible). Figure 7 shows two shapes with different topologies, a torus and a (*twisted*) cylinder, that have the same (inter)sections

with a set of (two) cutting planes. Hence, whatever is the reconstructed object from these sections, it would not be topologically consistent for at least one of these objects. In particular, the proposed reconstructed object ($\mathcal{R}$) is a torus which is not homotopy equivalent to the (*twisted*) cylinder ($\mathcal{O}$). In addition, we note that the Separation Condition may be verified for such a situation. Indeed, such a situation is exactly the case when the injective morphism between the fundamental groups of $\mathcal{O}$ and $\mathcal{R}$ is not surjective. This situation can be explained as follows: let x_1 and x_2 be two points in the sections S_1 and S_2 with the same lift x in $\mathrm{lift}(\mathcal{S}_\mathcal{C})$. The lift of any curve γ connecting x_1 and x_2 in $\mathcal{M}_\mathcal{C}$ provides a non-identity element of $\pi_1(\mathrm{lift}(\mathcal{S}_\mathcal{C}), x)$ which is not in the image of $\mathcal{L}_*$. We may avoid this situation with the following condition.

***Definition 10* (Intersection Condition).** — We say that the set of cutting planes verifies the Intersection Condition if for any pair of sections S_i and S_j in $\mathcal{S}_\mathcal{C}$, and for any connected component X of $\mathrm{lift}(S_i) \cap \mathrm{lift}(S_j)$, the following holds: there is a path $\gamma \subset \mathcal{M}_\mathcal{C}$ from a point $a \in S_i$ to a point $b \in S_j$ with $\mathrm{lift}(a) = \mathrm{lift}(b) = x \in X$ so that $\mathcal{L}_*(\gamma)$ is the identity element of $\pi_1(\mathrm{lift}(\mathcal{S}_\mathcal{C}), x)$, i.e., is contractible in $\mathrm{lift}(\mathcal{S}_\mathcal{C})$ with a homotopy respecting the base point x.

In Section 3.6, we will show how to verify the Intersection Condition. Let us first prove the surjectivity of the map $\mathcal{L}_*$ which is deduced directly from the Intersection Condition.

Theorem 7. — *Under the Intersection Condition, the induced map $\mathcal{L}_* : \pi_1(\mathcal{M}_\mathcal{C}) \to \pi_1(\mathrm{lift}(\mathcal{S}_\mathcal{C}))$ is surjective.*

Proof. — Let y_0 be a fixed point of $\mathcal{M}_\mathcal{C}$ and $x_0 = \mathcal{L}(y_0)$. We show that $\mathcal{L}_* : \pi_1\big(\mathcal{M}_\mathcal{C}, y_0\big) \to \pi_1\big(\mathrm{lift}(\mathcal{S}_\mathcal{C}), x_0\big)$ is surjective. Let α be a closed curve in $\mathrm{lift}(\mathcal{S}_\mathcal{C})$ which represents an element of $\pi_1\big(\mathrm{lift}(\mathcal{S}_\mathcal{C}), x_0\big)$. We show the existence of an element $\beta \in \pi_1\big(\mathcal{M}_\mathcal{C}, y_0\big)$ such that $\mathcal{L}_*(\beta) = [\alpha]$, where $[\alpha]$ denotes the homotopy class of α in $\pi_1\big(\mathrm{lift}(\mathcal{S}_\mathcal{C}), x_0\big)$. See Figure 8-left.
We can divide α into subcurves $\alpha_1, \ldots, \alpha_m$ such that α_j joins two points x_{j-1} and x_j, and is entirely in the lift of one of the sections S_j, for $j = 1, \ldots, m$. We may assume $y_0 \in S_1 = S_m$. For each $j = 1, \ldots, m$, let β_j be the curve in S_j joining two points z_j to w_j which is mapped to α_j under $\mathcal{L}$. Note that w_j and z_{j+1} (possibly) live in two different sections, but have the same image (x_j) under the lift map $\mathcal{L}$. Let X_j be the connected component of $\mathrm{lift}(S_j) \cap \mathrm{lift}(S_{j+1})$ which contains x_j, see Figure 8-right. According to the Intersection Condition, there is a path $\gamma_j \subset \mathcal{M}_\mathcal{C}$ connecting a point $a_j \in S_j$ to a point $b_{j+1} \in S_{j+1}$ such that $\mathrm{lift}(a_j) = \mathrm{lift}(b_{j+1}) = x'_j \in X_j$ and the image of γ_j under $\mathcal{L}$ is the identity element of $\pi_1(\mathrm{lift}(\mathcal{S}_\mathcal{C}), x'_j)$ (i.e., is contractible with a homotopy respecting the base point x'_j). Since X_j is connected, there is a path from x_j to x'_j in X_j, so lifting back this path to two paths from w_j to a_j in S_j and from b_{j+1} to z_{j+1} and taking the union of these two paths with γ_j, we infer the existence of a path $\gamma'_j \subset \mathcal{M}_\mathcal{C}$ connecting w_j to z_{j+1}, such that the image of γ'_j under $\mathcal{L}$ is contractible in $\mathrm{lift}(\mathcal{S}_\mathcal{C})$ with a homotopy respecting the base point x_j.

Let β be the path from x_0 to x_0 obtained by concatenating β_j and γ'_j alternatively, i.e., $\beta = \beta_1\gamma'_1\beta_2\gamma'_2 \ldots \beta_{m-1}\gamma'_m\beta_m\gamma'_m$. We claim that $\mathcal{L}_*([\beta]) = [\alpha]$. This is now easy to show: we have $\mathcal{L}_*(\beta) = \alpha_1\mathcal{L}_*(\gamma'_1)\alpha_2 \ldots \mathcal{L}_*(\gamma'_m)\alpha_m$, and all the paths $\mathcal{L}_*(\gamma'_j)$ are contractible to the constant path $[x_j]$ by a homotopy fixing x_j all the time. We deduce that under a homotopy fixing x_0, $\alpha_1\mathcal{L}_*(\gamma'_1) \ldots \mathcal{L}_*(\gamma'_m)\alpha_m$ is homotopic to $\alpha_1\alpha_2 \ldots \alpha_m = \alpha$, and this is exactly saying that $\mathcal{L}_*([\beta]) = [\alpha]$. The surjectivity follows. □

Putting together all the materials we have obtained, we infer the main theorem of this section.

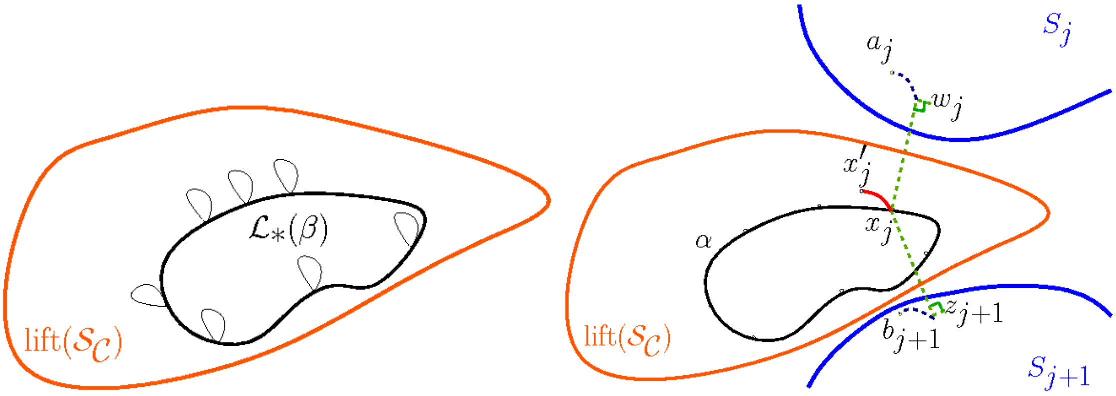

FIGURE 8. For the proof of Theorem 7: 3D illustration of two sections S_j and S_{j+1} (in blue) and their lift (in red). For any closed curve α in lift($\mathcal{S}_\mathcal{C}$), we show that there exists a closed curve β in $\mathcal{M}_\mathcal{C}$ so that the lift of β is homotopic to α in lift($\mathcal{S}_\mathcal{C}$). Left) We show the existence of β such that $\mathcal{L}_*(\beta)$ differs from α only in some contractible paths. Right) Notations used in the proof: We divide α into subcurves $\{\alpha_j\}_1^m$ such that α_j is entirely in the lift of the section S_j. By x_j we denote the point shared by α_j and α_{j+1} which is the lift of two points $w_j \in S_j$ and $z_{j+1} \in S_{j+1}$.

***Theorem 8* (Main Theorem-Part I).** — *Under the Separation and the Intersection Conditions, $\mathcal{R}_\mathcal{C}$ is homotopy equivalent to $\mathcal{O}_\mathcal{C}$ for any cell $\mathcal{C}$ of the arrangement.*

3.5. Generalized Nerve Theorem and Homotopy Equivalence of $\mathcal{R}$ and $\mathcal{O}$. — In this section, we extend the homotopy equivalence between $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$, in each cell $\mathcal{C}$, to a global homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$. To this end, we make use of a generalization of the nerve theorem. This is a folklore theorem and has been observed and used by different authors. For a modern proof of a still more general result, we refer to Segal's paper [**Seg68**]. (See also [**May03**], for a survey of similar results.)

***Theorem 9* (Generalized Nerve Theorem).** — *Let $H : X \to Y$ be a continuous map. Suppose that Y has an open cover $\mathcal{K}$ with the following two properties:*

- *Finite intersections of sets in $\mathcal{K}$ are in $\mathcal{K}$.*
- *For each $U \in \mathcal{K}$, the restriction $H : H^{-1}(U) \to U$ is a weak homotopy equivalence.*

Then H is a weak homotopy equivalence.

Let $H_\mathcal{C} : \mathcal{O}_\mathcal{C} \to \mathcal{R}_\mathcal{C}$ be the homotopy equivalence obtained in the previous sections between $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$. (So $H_\mathcal{C}$ is the composition of the retraction $\mathcal{O}_\mathcal{C} \to \mathcal{M}_\mathcal{C}$ and the inclusion $\mathcal{M}_\mathcal{C} \hookrightarrow \mathcal{R}_\mathcal{C}$.) Let $H : \mathcal{O} \to \mathcal{R}$ be the map defined by $H(x) = H_\mathcal{C}(x)$ if $x \in \mathcal{O}_\mathcal{C}$ for a cell $\mathcal{C}$ of the arrangement of the cutting planes. Note that H is well-defined since $H_\mathcal{C}|_{\mathcal{S}_\mathcal{C}} = id_{\mathcal{S}_\mathcal{C}}$, for all $\mathcal{C}$. In addition, since for all cell $\mathcal{C}$, $H_\mathcal{C}$ is continuous, H is continuous as well.
We can now apply the generalized nerve theorem by the following simple trick. Let ϵ be an infinitesimal positive value. For any cell $\mathcal{C}$ of the arrangement of the cutting planes, we define $\mathcal{O}_\mathcal{C}^\epsilon = \{x \in \mathbb{R}^3,\ d(x, \mathcal{O}_\mathcal{C}) < \epsilon\ \}$. Let us now consider the open covering $\mathcal{K}$ of $\mathcal{O}$ by these open sets and all their finite intersections. It is straightforward to check that for ϵ small enough, the restriction of H to each element of $\mathcal{K}$ is a weak homotopy equivalence. Therefore, according

to the generalized nerve theorem, H is a weak homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$. And by Whitehead's theorem, H is a homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$. Thus, we have

***Theorem 10* (Main Theorem-Part II).** — *Under the Separation and Intersection Conditions, the reconstructed object $\mathcal{R}$ is homotopy equivalent to the unknown original shape $\mathcal{O}$.*

3.6. How to Ensure the Intersection Condition?— In Section 2.2, we showed that the Separation Condition can be ensured with a sufficiently dense sample of cutting planes. In this section we provide a sufficient condition that implies the Intersection Condition.

We showed that by bounding from above the height of the cells by the reach of the object, we can ensure the Separation Condition. In order to ensure the Intersection Condition, we need a stronger condition on the height of the cells. As we will see, this condition is a *transversality* condition on the cutting planes that can be measured by the angle between the cutting planes and the normal to $\partial\mathcal{O}$ at contour-points.

***Definition 11* (Angle α_a).** — Let a be a point on the boundary of a section $A \in \mathcal{S}_\mathcal{C}$ on the plane P_A. We consider $m_i(a)$, that may be outside the cell $\mathcal{C}$. We define α_a as the angle between P_A and the normal to $\partial\mathcal{O}$ at a, i.e., $\alpha_a := \text{angle}(P_A, [a, m_i(a)])$, see Figure 9.

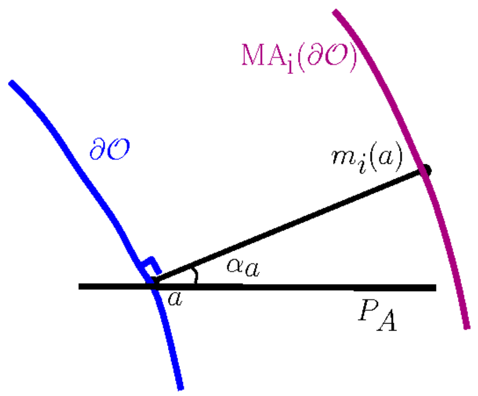

FIGURE 9. Definition of α_a (a 2D illustration).

Sufficient Conditions. — We now define the sampling conditions on the cutting planes. (See Section 2.3 for the definitions of $h_\mathcal{C}$ and $\text{reach}_\mathcal{C}(\mathcal{O})$.)

(C1) Density Condition : For any cell $\mathcal{C}$ of the arrangement, $h_\mathcal{C} < \text{reach}_\mathcal{C}(\mathcal{O})$.

(C2) Transversality Condition : For any cell $\mathcal{C}$,

$$h_\mathcal{C} < \frac{1}{2}\big(1 - \sin(\alpha_a)\big)\,\text{reach}(a), \forall\, a \in \partial\mathcal{S}_\mathcal{C}.$$

The Density Condition is based on the density of the sections. The Transversality Condition is defined in a way that the transversality of the cutting planes to $\partial\mathcal{O}$ and the distance between the sections are controlled simultaneously. (Indeed, the significance of $\sin(\alpha_a)$ is to control the transversality, and bounding from above $h_\mathcal{C}$ allows us to control the distance between the sections.)

Remark on the Transversality Condition. — The transversality of the cutting planes to $\partial\mathcal{O}$ seems to be a reasonable condition in practice, specially for applications in 3D ultrasound. Indeed, according to [**Rou03**, Section 1.2.1], from a technical point of view if a cut is not sufficiently transversal to the organ, the quality of the resulting 2D ultrasonic image is not acceptable for diagnosis.

According to Lemma 2 of Section 2.2, the Density Condition implies the Separation Condition. We will show that under the Transversality Condition, the Intersection Condition is verified. Therefore, by increasing the density of the sections of $\mathcal{O}$, with preferably transversal cutting planes, we can ensure the required sampling conditions, and as a consequence, provide a topologically consistent reconstruction of $\mathcal{O}$.

Theorem 11 **(Main Theorem-Part III)**. — *If the set of cutting planes verifies the* Density *and the* Transversality Conditions*, then the* Separation *and the* Intersection Conditions *are verified. Therefore, the proposed reconstructed object $\mathcal{R}$ is homotopy equivalent to the unknown original shape $\mathcal{O}$.*

3.7. *Proof of Theorem 11.* — This section is devoted to the proof of Theorem 11. It remains to show that under the Transversality Condition, the Intersection Condition is verified.

We need the following notations.

Notation $\big(K_i(\mathcal{S_C})$ **and** $K_e(\mathcal{S_C})\big)$**.** — Recall that VorSkel($\mathcal{C}$) is the locus of the points with more than one nearest point in $\partial\mathcal{C}$. We write $K_i(\mathcal{S_C})$ (resp. $K_e(\mathcal{S_C})$) for the set of points $x \in$ VorSkel($\mathcal{C}$) such that all the nearest points of x in $\partial\mathcal{C}$ lie inside (resp. outside) the sections.

Notation $\big(\overline{m}_i(a)$ **and** $\overline{m}_e(a)\big)$**.** — Let a be a point on the boundary of a section $A \in \mathcal{S_C}$ on the plane P_A. We write $\overline{m}_i(a)$ (resp. $\overline{m}_e(a)$) for the orthogonal projection of $m_i(a)$ (resp. $m_e(a)$) onto P_A. See Figure 10. Note that we have $d(m_i(a), \overline{m}_i(a)) = \sin(\alpha_a)\, d(a, m_i(a))$ and $d(m_e(a), \overline{m}_e(a)) = \sin(\alpha_a)\, d(a, m_e(a))$.

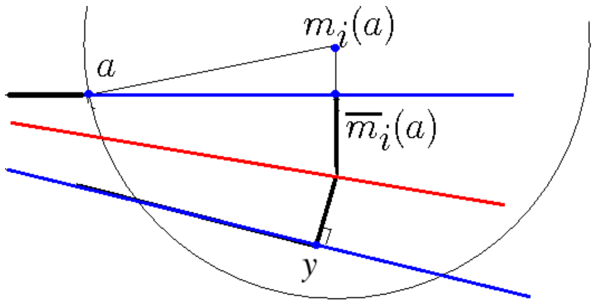

FIGURE 10. Definition of $\bar{m}_i(a)$ (a 2D illustration).

Lemma 3. — If the Transversality Condition is verified, then for any $a \in \partial\mathcal{S_C}$, we have $\mathrm{lift}(\overline{m}_i(a)) \in K_i(\mathcal{S_C})$ and $\mathrm{lift}(\overline{m}_e(a)) \in K_e(\mathcal{S_C})$. In addition, the two segments that join $\mathrm{lift}(\overline{m}_i(a))$ to its nearest points in $\partial\mathcal{C}$ both lie entirely in $\mathcal{M_C}$.

Proof. — We first show that $\mathrm{lift}(\overline{m}_i(a)) \in K_i(\mathcal{S}_\mathcal{C})$. By symmetry, the analogue property for $\mathrm{lift}(\overline{m}_e(a))$ can be proved similarly. To simplify the notation in the proof of this lemma, we write m_i for $m_i(a)$, and $\overline{m}_i$ for $\overline{m}_i(a)$, and simply write $B(m_i)$ for the ball centered at m_i of radius $d(m_i, a)$. We have $d(m_i, \overline{m}_i) \leq d(m_i, a)$. Thus, $\overline{m}_i$ lies in $B(m_i)$. As this ball is contained in $\mathcal{O}$, $\overline{m}_i$ is in $\mathcal{O}$. Consider now $\mathrm{lift}(\overline{m}_i)$ on $\mathrm{VorSkel}(\mathcal{C})$, and call y the point distinct from $\overline{m}_i$ such that $\mathrm{lift}(\overline{m}_i) = \mathrm{lift}(y)$, see Figure 10. To have $\mathrm{lift}(m_i) \in K_i(\mathcal{S}_\mathcal{C})$, we need to show that y is in $\mathcal{O}$. We have

$$\begin{aligned} d(m_i, y) \ &\leq \ d(m_i, \overline{m}_i) + d(\overline{m}_i, \mathrm{lift}(\overline{m}_i)) + d(\mathrm{lift}(y), y) \\ &\leq \ \sin(\alpha_a)\, d(a, m_i) + 2\, h_\mathcal{C} \ \leq \ d(a, m_i). \end{aligned}$$

Thus, y belongs to $B(m_i)$, and we deduce that $y \in \mathcal{O}$ and $\mathrm{lift}(\overline{m}_i) \in K_i(\mathcal{S}_\mathcal{C})$. In addition, the ball B centered at $\mathrm{lift}(\overline{m}_i)$ which passes through $\overline{m}_i$ and y is entirely contained in $B(m_i) \subset \mathcal{O}$. Thus, the interior of B is empty of points of $\partial\mathcal{O}_\mathcal{C}$ and B is a medial ball of $\mathcal{O}_\mathcal{C}$, and its center $\mathrm{lift}(\overline{m}_i)$ belongs to $\mathrm{MA}_i(\partial\mathcal{O}_\mathcal{C})$. Since $\overline{m}_i$ and y are in $\mathcal{S}_\mathcal{C}$, according to the definition of $\mathcal{M}_\mathcal{C}$, the line-segments $[\mathrm{lift}(\overline{m}_i), \overline{m}_i]$ and $[\mathrm{lift}(\overline{m}_i), y]$ lie entirely in $\mathcal{M}_\mathcal{C}$. □

Let S_a and S_b be two sections in $\mathcal{S}_\mathcal{C}$ such that $\mathrm{lift}(S_a) \cap \mathrm{lift}(S_b)$ is non-empty. (As it will become clear, a and b will be two chosen points in S_a and S_b respectively.) We recall that $\mathcal{L}_*$ denotes the homomorphism $\mathcal{L}_* : \pi_j(\mathcal{M}_\mathcal{C}) \to \pi_j(\mathrm{lift}(\mathcal{S}_\mathcal{C}))$ induced by the lift function $\mathcal{L} : \mathcal{M}_\mathcal{C} \to \mathrm{lift}(\mathcal{S}_\mathcal{C})$. We will show that for any two points $a \in \partial S_a, b \in \partial S_b$ such that $\mathrm{lift}(a) = \mathrm{lift}(b)$, there exists a path $\gamma \subset \mathcal{M}_\mathcal{C}$ between a and b so that $\mathcal{L}_*(\gamma)$ is contractible in $\mathcal{L}_*(\pi_1(\mathcal{M}_\mathcal{C}))$. Let P_a and P_b be the cutting-planes of S_a and S_b respectively. One of the two following cases happen:

- If the segment $[a, \overline{m}_i(a)]$ (i.e., the projection of the segment $[a, m_i(a)]$ onto P_a) is not cut by any other cutting plane, then we claim that $\mathrm{lift}(a)$ is connected to $\mathrm{lift}(\overline{m}_i(a))$ in $K_i(\mathcal{S}_\mathcal{C})$. Consider the two-dimensional ball B in the plane P_a centered at $\overline{m}_i(a)$ and passing through a. This ball is contained in the 3D ball centered at $m_i(a)$ and passing through a, which lies entirely inside $\mathcal{O}$. Considering the intersection of this 3D ball with P_a, we infer that B lies in the section S_a. Therefore, $\mathrm{lift}(B)$ is entirely contained in $\mathrm{lift}(S_a)$. In Figure 11, $\mathrm{lift}(B)$ is colored in green.

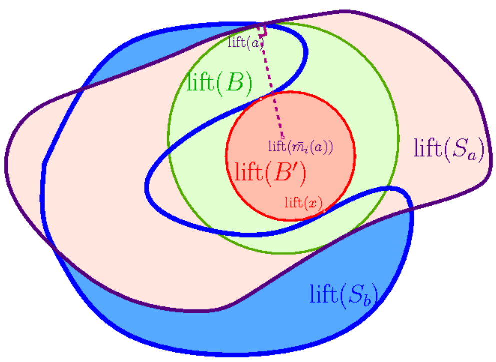

FIGURE 11. For the proof of Theorem 11: the lift of two sections S_a and S_b on $\mathrm{VorSkel}(\mathcal{C})$ is illustrated. We prove that the lift of the segment $[a, \bar{m}_i(a)]$ lies in $\mathrm{lift}(S_a) \cap \mathrm{lift}(S_b)$.

Since $\text{lift}(a) = \text{lift}(b)$, we have $\text{lift}(a) \in \text{lift}(S_a) \cap \text{lift}(S_b)$. On the other hand, according to Lemma 3, $\text{lift}(\overline{m}_i(a))$ lies in $\text{lift}(S_a) \cap \text{lift}(S_b)$. For the sake of a contradiction, suppose that $\text{lift}(a)$ is not connected to $\text{lift}(\overline{m}_i(a))$ in $K_i(\mathcal{S}_\mathcal{C})$. In this case, $\text{lift}(B)$ intersects (at least) two different connected components of $\text{lift}(S_a) \cap \text{lift}(S_b)$, see Figure 11. Since $\text{lift}(B) \subset \text{lift}(S_a)$, we deduce that $\text{lift}(B)$ intersects $\text{lift}(S_b)$ in (at least) two different connected components. Consider now the maximal open ball contained in $\text{lift}(B)$ which is empty of points of $\text{lift}(S_b)$. Such a ball consists of the lift of a 2D ball B' in the plane P_b which is tangent to ∂S_b at two points x and x'. B' is the intersection of the 3D medial ball of the complementary set of $\mathcal{O}$ passing through x and x'. Thus, $\overline{m}_e(x)$, which is the projection of the center of this 3D ball ($m_e(x)$) onto P_b, lies in B'. We have $\text{lift}(\overline{m}_e(x)) \in \text{lift}(B') \subseteq \text{lift}(B) \subset \text{lift}(S_a)$. Thus, one of the nearest points of $\text{lift}(\overline{m}_e(x))$ in $\partial\mathcal{C}$ lies in $S_a \subset \mathcal{S}_\mathcal{C}$. This contradicts $\text{lift}(\overline{m}_e(a)) \in K_e(\mathcal{S}_\mathcal{C})$ (Lemma 3).

Therefore, $\text{lift}(a)$ is connected to $\text{lift}(\overline{m}_i(a))$ in $K_i(\mathcal{S}_\mathcal{C})$. Let us call a' and b' the nearest points of $\text{lift}(\overline{m}_i(a))$ in S_a and S_b respectively, see Figure 12-left. According to Lemma 3, the line-segments $[a', \text{lift}(\overline{m}_i(a))]$ and $[b', \text{lift}(\overline{m}_i(a))]$ lie inside $\mathcal{M}_\mathcal{C}$. We now define the path γ, colored in red in Figure 12-left, as the concatenation of four line-segments: $[a, a'] \subset S_a$, $[a', \text{lift}(\overline{m}_i(a))]$, $[b', \text{lift}(\overline{m}_i(a))]$ and $[b', b] \subset S_b$. We know that $[a', \text{lift}(\overline{m}_i(a))]$ and $[b', \text{lift}(\overline{m}_i(a))]$ are mapped to a single point $\text{lift}(\overline{m}_i(a))$ by the lift function. Thus, the image of γ under the lift function is the line-segment $[\text{lift}(a), \text{lift}(\overline{m}_i(a))]$, which is trivially contractible in $\mathcal{L}_*(\pi_1(\mathcal{M}_\mathcal{C}))$.

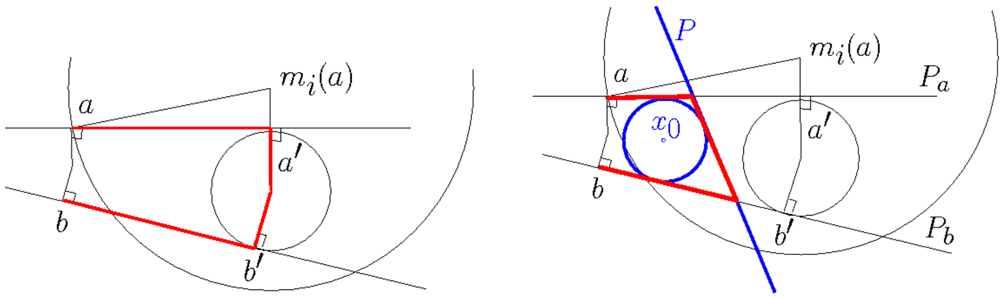

FIGURE 12. A 2D illustration of the two cases considered in the proof of Theorem 11: the path between a and b in $\mathcal{M}_\mathcal{C}$ is given in red.

• If the segment $[a, \overline{m}_i(a)]$ (i.e., the projection of the segment $[a, m_i(a)]$ onto P_a) is cut by a cutting plane P (Figure 12 (Right)), then we claim that a is connected to b in $\mathcal{S}_\mathcal{C}$. Let l be the lift of the segment $[a, \overline{m}_i(a)]$. For any point $x \in l$, we write $B(x)$ for the open ball centered at x and tangent to the planes of sections S_a and S_b. Since $\text{lift}(a) \in \text{lift}(S_a) \cap \text{lift}(S_b)$, $B(\text{lift}(a))$ is empty of points of $\partial\mathcal{C}$. In particular, it does not intersect P. Hence, there exists a point $x_0 \in l$, such that P is tangent to $B(x_0)$. See Figure 12 (Right). We also observe that P intersects $[a, \overline{m}_i(a)]$ and the corresponding segment in S_b which is also mapped to l by the lift function (the segment $[b, b']$ in Figure 12 (Right)). In this case, since $\mathcal{S}_\mathcal{C} \subset \mathcal{M}_\mathcal{C}$, there is a path in $\mathcal{S}_\mathcal{C}$ (and so in $\mathcal{M}_\mathcal{C}$) between a and b, colored in red in Figure 12 (Right), such that its image by the lift function is the concatenation of $[\text{lift}(a), x_0]$ with a contractible path from x_0 to x_0.

This ends the proof of Theorem 11.

3.8. Deforming the Homotopy Equivalence to a Homeomorphism. — Using the homotopy equivalence between $\mathcal{R}$ and $\mathcal{O}$, one can show the following theorem.

***Theorem 12* (Main Theorem-Part IV).** — *Under the Separation and the Intersection Conditions, the two topological manifolds $\mathcal{R}$ and $\mathcal{O}$ are homeomorphic (in addition, they are isotopic).*

Although this result is stronger than the homotopy equivalence, the way our proof works makes essentially use of the topological study of the previous sections.

Proof. — Again, we first argue in each cell of the arrangement and show the existence of a homeomorphism between $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ whose restriction to $\mathcal{S}_\mathcal{C}$ is the identity map. Gluing these homeomorphisms together, one obtains a global homeomorphism between $\mathcal{R}$ and $\mathcal{O}$. Let $\mathcal{C}$ be a cell of the arrangement of the cutting planes. A similar method used to prove the homotopy equivalence between $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ shows that $\partial\mathcal{R} \cap \mathcal{C}$ and $\partial\mathcal{O} \cap \mathcal{C}$ are homotopy equivalent 2-manifolds and are therefore homeomorphic, and in addition there exists a homeomorphism $\beta_\mathcal{C} : \partial\mathcal{O} \cap \mathcal{C} \to \partial\mathcal{R} \cap \mathcal{C}$ which induces identity on the boundary of sections in $\mathcal{S}_\mathcal{C}$. We showed that the topology of $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ is completely determined by their fundamental groups, i.e., all the higher homotopy groups of $\mathcal{R}_\mathcal{C}$ and $\mathcal{O}_\mathcal{C}$ are trivial. Moreover, there is an isomorphism between $\pi_1(\mathcal{O}_\mathcal{C})$ and $\pi_1(\mathcal{R}_\mathcal{C})$, and the induced map $(\beta_\mathcal{C})_* : \pi_1(\partial\mathcal{O} \cap \mathcal{C}) \to \pi_1(\partial\mathcal{R} \cap \mathcal{C})$ on first homotopy groups is consistent with this isomorphism (in the sense that there exists a commutative diagram of first homotopy groups). This shows that there is no obstruction in extending $\beta_\mathcal{C}$ to a map $\alpha_\mathcal{C} : \mathcal{O}_\mathcal{C} \to \mathcal{R}_\mathcal{C}$ inducing the corresponding isomorphism between $\pi_1(\mathcal{O}_\mathcal{C})$ and $\pi_1(\mathcal{R}_\mathcal{C})$, and such that the restriction of $\alpha_\mathcal{C}$ to $\mathcal{S}_\mathcal{C}$ remains identity. Since all the higher homotopy groups of $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ are trivial, it follows that $\alpha_\mathcal{C}$ is a homotopy equivalence. We can now apply the following theorem due to Waldhausen which shows that α can be deformed to homeomorphism between $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ by a deformation which does not change the homeomorphism $\alpha_\mathcal{C}$ between the boundaries. A compact 3-manifold M is called *irreducible* if $\pi_2(M)$ is trivial. We note that $\mathcal{O}_\mathcal{C}$ and $\mathcal{R}_\mathcal{C}$ are irreducible.

Theorem 13* (Waldhausen).** — *Let $f : M \to M'$ be a homotopy equivalence between orientable irreducible 3-manifolds with boundaries such that f takes the boundary of M onto the boundary of M' homeomorphically. Then f can be deformed to a homeomorphism $M \to M'$ by a homotopy which is fixed all the time on the boundary of M. (See* **[Mat03], page 220, for a proof.)*

Applying Waldhausen's theorem, one obtains a homeomorphism $\tilde{\alpha}_\mathcal{C}$ from $\mathcal{O}_\mathcal{C}$ to $\mathcal{R}_\mathcal{C}$ which is identity on the sections in $\mathcal{S}_\mathcal{C}$. Gluing $\tilde{\alpha}_C$, one obtains a global homeomorphism form $\mathcal{O}$ to $\mathcal{R}$. Moreover, according to Chazal and Cohen-Steiner's work **[CCS05]** (Corollary 3.1), since $\mathcal{R}$ and $\mathcal{O}$ are homeomorphic and $\mathcal{R}$ contains the medial axis of $\mathcal{O}$, $\mathcal{R}$ is isotopic to $\mathcal{O}$. □

Conclusion. — In this paper, we presented the first topological studies in shape reconstruction from cross-sectional data. We showed that the generalization of the classical overlapping criterion to solve the correspondence problem between unorganized cross-sections, proposed by Liu et al. in **[LBD⁺08]**, preserves the homotopy type of the shape under some appropriate sampling conditions. In addition, we proved that in this case, the homotopy equivalence between the reconstructed object and the original shape can be deformed to a homeomorphism. Even, more strongly, the two objects are isotopic.

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A
Homotopy Theory Basics

In this section, we briefly recall some basic definitions that are used in the paper.

***Definition 12* (Homotopy).** — A *homotopy* between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H : X \times [0,1] \to Y$ such that for all points $x \in X$, $H(x,0) = f(x)$ and $H(x,1) = g(x)$. f is said to be *homotopic* to g if there exists a homotopy between f and g.

***Definition 13* (Homotopy Equivalence).** — Two topological spaces X and Y are *homotopy equivalent* or of the *same homotopy type* if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to id_Y.

***Definition 14* (Homotopy Groups, Fundamental Group).** — Let X be a space with a base point $x_0 \in X$. Let S^i denote the i-sphere for a given $i \geq 1$, in which we fixed a base point b. The i-dimensional homotopy group of X at the base point x_0, denoted by $\pi_i(X, x_0)$, is defined to be the set of homotopy classes of maps $f : S^i \to X$ that map the base point b to the base point x_0.

Thus if X is path-connected, the group $\pi_i(X, x_0)$ is, up to isomorphism, independent of the choice of base point x_0. In this case the notation $\pi_i(X, x_0)$ is often abbreviated to $\pi_i(X)$.

Let X be a path-connected space. The first homotopy group of X, $\pi_1(X)$, is called the *fundamental group* of X.

***Definition 15* (Simply-Connected Spaces).** — The path-connected space X is called *simply-connected* if its fundamental group is trivial.

***Definition 16* (Weak Homotopy Equivalence).** — A map $f : X \to Y$ is called a weak homotopy equivalence if the group homomorphisms induced by f on the corresponding homotopy groups, $f_* : \pi_i(X) \to \pi_i(Y)$, for $i \geq 0$, are all isomorphisms. It is easy to see that any homotopy equivalence is a weak homotopy equivalence, but the inverse is not necessarily true. However, Whitehead's Theorem states that the inverse is true for maps between CW-complexes.

***Theorem 14* (Whitehead's Theorem).** — If a map $f : X \to Y$ between connected CW-complexes induces isomorphisms $f_* : \pi_i(X) \to \pi_i(Y)$ for all $i \geq 0$, then f is a homotopy equivalence.

***Definition 17* ((Strong) Deformation Retract).** — Let X be a subspace of Y. A homotopy $H : Y \times [0,1] \to Y$ is said to be a (strong) deformation retract of Y onto X if:

- For all $y \in Y$, $H(y,0) = y$ and $H(y,1) \in X$.
- For all $x \in X, H(x,1) = x$.
- (and for all $x \in X, H(x,t) = x$.)

***Definition 18* (Homeomorphism).** — Two topological spaces X and Y are *homeomorphic* if there exists a continuous and bijective map $h : X \to Y$ such that h^{-1} is continuous. the map h is called a homeomorphism from X to Y.

***Definition 19* (Isotopy).** — Two topological spaces X and Y embedded in $\mathbb{R}^d$ are *isotopic* if there exists a continuous map $i : [0,1] \times X \to \mathbb{R}^d$ such that $i(0,.)$ is the identity over X, $i(1,X) = Y$ and for any $t \in [0,1]$, $i(t,.)$ is a homeomorphism from X to its image. The map i is called an isotopy from X to Y.

Let us recall the definition of the universal cover, and refer to classical books in topology for more details.

***Definition 20* (Universal Cover).** — Let X be a topological space. A *covering space* of X is a space C together with a continuous surjective map $\phi : C \to X$ such that for every $x \in X$, there exists an open neighborhood U of x, such that $\phi^{-1}(U)$ is a disjoint union of open sets in C, each of which is mapped homeomorphically onto U by ϕ. A connected covering space is called a *universal cover* if it is simply connected.

The universal cover exists and is unique up to homeomorphism.

***Lemma 4* (Lifting Property of the Universal Cover).** — Let X be a (path-) connected topological space and $\widetilde{X}$ be its universal cover, and $\phi : \widetilde{X} \to X$ be the map given by the covering. Let Y be any simply connected space, and $f : Y \to X$ be a continuous map. Given two points $\tilde{x} \in \widetilde{X}$ and $y \in Y$ with $\phi(\tilde{x}) = f(y)$, there exists a unique continuous map $g : Y \to \widetilde{X}$ so that $\phi \circ g = f$ and $\phi(y) = \tilde{x}$. This is called the *lifting property* of $\widetilde{X}$.

Since all the spheres S_i of dimensions $i \geq 2$ are simply connected, we may easily deduce.

***Corollary 1*.** — For any (path-) connected space X with universal cover $\tilde{X}$, we have $\pi_i(\tilde{X}) = \pi_i(X)$ for all $i \geq 2$.

We now recall the definition of the Hurewicz map $h_i : \pi_i(X) \to H_i(X)$. For an element $[\alpha] \in \pi_i(X)$ presented by $\alpha : S^i \to X$, $h_i([\alpha])$ is defined as the image of the fundamental class of S^i in $H_i(S^i)$ under the map $\alpha_* : H_i(S^i) \to H_i(X)$, i.e., $h_i([\alpha]) = \alpha_*(1)$.

***Theorem 15* (Hurewicz Isomorphism Theorem).** — The first non-trivial homotopy and homology groups of a simply-connected space occur in the same dimension and are isomorphic. In other words, for X simply connected, the Hurewicz map $h_i : \pi_i(X) \to H_i(X)$ is an isomorphism for the first i with π_i (or equivalently H_i) non-trivial.